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REFINEMENT OF A PRECONDITIONED ACCELERATED OVER-RELAXATION METHOD FOR SOLUTION OF LINEAR ALGEBRAIC SYSTEM

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Abstract
In Linear Algebra system, a significant improvement of the iteration matrix will reduce the spectral radius and accelerate the rate of convergence of the specific method while solving system of linear equations of the form $Ax = b$. This prompt us to refine the Preconditioned Accelerated Over Relaxation (PAOR) method called Refinement of Preconditioned Accelerated Over-Relaxation (RPAOR) so as to accelerate the convergence rate of the method. A refinement of Preconditioned Accelerated Over Relaxation method that would minimize the spectral radius, when compared to AOR and PAOR method is proposed in this paper. We investigated the convergence of the method and presented some numerical examples to check the performance of the method. The results indicate the superiority of the method over some existing methods

Keywords: Preconditioned, Accelerated, Over Relaxation, Convergence, Spectral radius

Introduction
Numerical analysis is the area of mathematics and computer science that creates, analyses, and implements algorithms for solving numerically the problems of continuous mathematics. Such problems originate generally from real-world applications of algebra, geometry and calculus, and they involve variables which vary continuously. These problems occur throughout the natural sciences, social science, engineering, medicine, and business.

The solution of system of linear equations can be accomplished by a numerical method which falls in one of two categories: direct or iterative methods employed to solve such linear systems and utilizing some properties of the coefficient matrices like sparseness for example usually makes it possible for storage reduction, run time and cost (Kiusalaas, 2005). Iterative methods are quite effective and usually preferable over direct methods in finding solutions to sparse/large linear systems. Modification of existing iterative methods in order to achieve higher rate of convergence led to several refinement methods. Dafchahi (2008) introduced the refinement of Jacobi (RJ) method and proved that the method is superior to Jacobi method. Vatti & Eneyew (2011) and Genanew (2016) enhanced the convergence rate of the Gauss-Seidel method by developing the refinement of Gauss-Seidel (RGS) method. Kyurkhiev & Iliev (2013) modified the SOR and SSOR schemes by proposing the refinement of Successive Over relaxation (RSOR) and refinement of Symmetric Successive Over Relaxation (RSSOR) methods. Vatti et al. (2015) improved on the SOR scheme and proposed the RSOR method. Vatti et al. (2018) attempted to magnify convergence rate of the AOR method by the method called Refinement of Accelerated Over-Relaxation (RAOR) method. Muleta & Gofe (2018) developed a Refinement of Generalized Accelerated Over Relaxation (RGAOR) method. Audu et al. (2021a) introduced the Extended Accelerated Over Relaxation method, an efficient three-parameter method that generalizes the AOR, SOR, Jacobi and Gauss-Seidel methods. Audu et al., (2021b) refined the Extended Accelerated Over Relaxation (EAOR) method into a more efficient method that would hasten the numerical solution of linear systems. Abdullahi & Muhammad (2021) refined the work of Ndanusa & Adeboye (2012) called Refinement of Preconditioned Successive Over-relaxation (SOR) algorithm for the solution of linear Algebraic system. Also, Ndanusa et al., (2021) refined the work of Abdullahi & Ndanusa (2020) called Refinement of Preconditioned Accelerated Over Relaxation (AOR) iterative method that gives the faster convergence to the linear systems.
This present work aims to undertake iterative refinement of (Ismaila et al., 2023) PAOR, into a more efficient method that would hasten the numerical solution of linear system.

Material and Method

$$Bt = b, \tag{1}$$

where $B = (b_{ij}) \in \mathbb{R}^{n \times n}$ is a non singular matrix with non vanishing diagonal elements, and where $t \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ are respectively vectors of unknown and pre assigned variables. Some methods of preconditioning which improve the rate of convergence of these iterative methods have been developed. Recall the usual splitting of the coefficient matrix $B \in \mathbb{R}^{n\times n}$,

$$B = D_B - L_B - U_B \qquad (2)$$

Where $D_B = diag(b_{11}, b_{22}, \cdots, b_{nn})$ is the diagonal part of B, $-L_B$ and $-U_B$ its strictly lower and strictly upper parts, respectively. If $b_{ii} \neq 0$ for all $i \in \mathbb{N}$ ($i = 1,2,\cdots,n$), then we can multiply the linear system (1) by ${D_B}^{-1}$, arising therefrom the splitting of the matrix

$$B = I - L - U \qquad (3)$$

where $I = {D_B}^{-1}D_B, L = {D_B}^{-1}L_B$ and $U = {D_B}^{-1}U_B$.
Suppose $B = P - Q$ is a regular splitting of the coefficient matrix $B = (b_{ij})$, and then the basic iterative method for the solution of system (1) can be expressed in the form

$$t^{(i+1)} = P^{-1}Qx^{(k)} + P^{-1}b, \qquad i = 0,1,2,\cdots \quad (4)$$

where $P^{-1}Q$ is known as the iteration matrix of the method. The iteration (4) is known to converge to the exact solution $t = B^{-1}b$ for any initial vector value $t^{(0)} \in \mathbb{R}^n$ if and only if the spectral radius $\rho(P^{-1}Q) < 1$. The smaller the spectral radius the faster the convergence speed of the iterative method. The classical AOR iterative method given in Hadjidimos (1978) for solving (1) is defined as

$$t^{(i+1)} = \ell_{r,\omega}t^{(i)} + (I - rL)^{-1}\omega b \qquad i = 0,1,2,\cdots \quad (5)$$

with the iteration matrix $\mathcal{L}_{r,\omega}$ given as

$$\ell_{r,\omega} = (I - rL)^{-1}[(1-\omega)I + (\omega - r)L + \omega U] \qquad (6)$$

where ω and r are real parameters with $\omega \neq 0$ and
$P = (I - rL)$ & $Q = [(1-\omega)I + (\omega - r)L + \omega U]$

Also, in formulation of refinement iterative, we consider a usual splitting of the form;

$$A = \frac{1}{\omega}(P - Q), \quad \text{which implies that} \quad \omega A = (P - Q) \qquad (7)$$

$$At = b \qquad (8)$$

$$\omega At = \omega b$$

$$\omega\left(\frac{1}{\omega}(P-Q)\right)t = \omega b$$

$$(P-Q)t = \omega b$$

(9)

Substitute P and Q in equation (9)

$$\{(I-rL)-[(1-\omega)I+(\omega-r)L+\omega U]\}t = \omega b$$

(10)

$$(I-rL)t-[(1-\omega)I+(\omega-r)L+\omega U]t = \omega b$$

$$(I-rL)t = [(1-\omega)I+(\omega-r)L+\omega U]t+\omega b$$

$$= [(I-rL)-(\omega-\omega L-\omega U)]t+\omega b$$

$$= (I-rL)t-\omega(I-L-U)t+\omega b$$

$$= (I-rL)t-\omega At+\omega b$$

$$(I-rL)t = (I-rL)t+\omega(b-At)$$

$$t = t+\omega(I-rL)^{-1}(b-At)$$

(11)

From equation (11) the Refinement of Accelerated Over Relaxation formula takes the form.

$$\bar{t}^{(i+1)} = t^{(i+1)}+\omega(I-rL)^{-1}(b-At^{(i+1)}) \tag{12}$$

Where $t^{(i+1)}$ appearing in the right hand side, is $(i+1)^{th}$ estimation of the AOR iterative method. Using the P and Q to obtain the required iteration in (18)

$$\bar{t}^{(i+1)} = t^{(i+1)}+\omega(I-rL)^{-1}(b-At^{(i+1)})$$

$$\bar{t}^{(i+1)} = t^{(i+1)}+\omega P^{-1}(b-At^{(i+1)})$$

$$\bar{t}^{(i+1)} = t^{(i+1)}+\omega P^{-1}b-\omega P^{-1}At^{(i+1)}$$

$$\bar{t}^{(i+1)} = t^{(i+1)}+\omega P^{-1}b-P^{-1}(P-Q)t^{(i+1)}$$

$$= t^{(i+1)}+\omega P^{-1}b-P^{-1}Pt^{(i+1)}+P^{-1}Qt^{(i+1)}$$

$$= t^{(i+1)}+\omega P^{-1}b-P^{-1}Pt^{(i+1)}+P^{-1}Qt^{(i+1)}$$

$$= \omega P^{-1}b+P^{-1}Qt^{(i+1)}$$

$$\bar{t}^{(i+1)} = P^{-1}\omega b + P^{-1}Qt^{(i+1)} \qquad (13)$$

But $t^{(i+1)} = P^{-1}Qt^{i} + P^{-1}\omega b,$ put to equation (13) to obtain

$$\bar{t}^{(i+1)} = P^{-1}\omega b + P^{-1}Q(P^{-1}Qt^{i} + P^{-1}\omega b) \qquad (14)$$

$$= P^{-1}\omega b + [P^{-1}Q]^{2}t^{i} + P^{-1}QP^{-1}\omega b$$

$$= P^{-1}\omega b + [P^{-1}Q]^{2}t^{i} + [P^{-1}]^{2}Q\omega b$$

$$\bar{t}^{(i+1)} = [P^{-1}Q]^{2}t^{i} + P^{-1}[P^{-1}Q + 1]\omega b \qquad (15)$$

But $P = (I - rL)$ and $Q = [(1-\omega)I + (\omega - r)L + \omega U]$

$$\bar{t}^{(i+1)} = \left\{(I - rL)^{-1}[(1-\omega)I + (\omega - r)L + \omega U]\right\}^{2} t^{i}$$
$$+ (I - rL)^{-1}[(I - rL)^{-1}((1-\omega)I + (\omega - r)L + \omega U) + 1]\omega b \qquad (16)$$

The method in (16) is called the RAOR method,

$$\bar{t}^{(i+1)} = \ell_{r,\omega}^{2} + (I - rL)^{-1}[\ell_{r,\omega} + (I - rL)^{-1}]\omega b \qquad (17)$$

T hat is

$$\bar{t}^{(i+1)} = R_{\ell_{r,\omega}} t^{i} + d, \quad n = 0,1,2,\ldots \qquad (18)$$

Where

$R_{\ell_{r,\omega}} = \left\{(I - rL)^{-1}[(1-\omega)I + (\omega - r)L + \omega U]\right\}^{2}$ is the iteration matrix of the RPAOR method and $d = (I - rL)^{-1}[(I - rL)^{-1}((1-\omega)I + (\omega - r)L + \omega U) + 1]\omega b$. The spectral radius of the RAOR method is the largest eigen value of its iteration matrix represented as $\rho(RAOR)$

Besides the iteration matrix and parameters involved in the iterative methods, convergence is also dependent on the nature of the linear systems of equations themselves. Therefore, in order to improve efficiency of the iterative method (18), the linear system (8) is transformed into the equivalent preconditioned system

$$\mathcal{P}At = \mathcal{P}b \qquad (19)$$

where $\mathcal{P}$ is a nonsingular matrix called a preconditioner. Different preconditioners have been advanced by several researchers for the preconditioned system (). Gunawardena *et al*. (1991), Kohno *et al.* (1997), Evans *et al.* (2001), Li *et al.* (2000), Li *et al.* (2007), Wu *et al.* (2007), Wu and Huang (2007), Yun and Kim (2008), Wang and Song (2009), Darvishi *et al.* (2011), Li (2011), Ndanusa & Adeboye (2012), Huang *et al.* (2016), Behzadi (2019) and Wang (2019) Abdullahi and Ndanusa (2021) are some instances of application of the preconditioned system (19) to improve the convergence of the AOR method. The preconditioner of Ismaila et al., (2023) is applied to system (19) to obtain the iterative system of the form,

$$\ell_{r,\omega}^{2} = \left\{(\widetilde{D} - r\widetilde{L})^{-1}[(1-\omega)\widetilde{D} + (\omega - r)\widetilde{L} + \omega\widetilde{U}]\right\}^{2} \quad (20)$$

where $\widetilde{D} = I + D_1$, $\tilde{L} = L + \tilde{L}_{\tilde{S}} + L_1$, $\widetilde{U} = U + U_{\tilde{S}} + U_1$, $\tilde{S} = -L_{\tilde{S}} - U_{\tilde{S}}$, and $-\tilde{S}L - \tilde{S}U = D_1 - L_1 - U_1$; where D_1, $-L_1$, and $-U_1$ are the diagonal, strictly lower and strictly upper parts of $-\tilde{S}L - \tilde{S}U$ respectively; and $-L_{\tilde{S}}$ and $-U_{\tilde{S}}$ are the strictly lower and strictly upper parts of $\tilde{s}$ respectively. Therefore, equation (20) is now the refinement of preconditioned AOR (RPAOR) iterative matrix

Convergence Analysis of the Method

Definition 1: A square matrix A is considered as a weak diagonally dominant if and only if $|a_{ii}| \geq \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad i = 1,2,\ldots,n$

Definition 2: A square matrix $(A_{ij})_{p,q}$ is considered as an L matrix if $a_{ij} \leq 0\, (i \neq j)$ and $a_{ii} > 0,\ i = 1,2,\ldots,n$

Definition 3: Spectral radius is the absolute value of the greatest eigenvalues of $p \times p$ matrix denoted as $\rho(A) = \max_{\lambda_i \in A} |\lambda_i|$

Definition 4: A square matrix $(A_{ij})_{p,q}$ is considered as an M matrix if $a_{ij} \leq 0,\ a_{ii} > 0,\ i = 1,2,\ldots,n, A$ is non-singular and $A^{-1} \geq 0$.

Definition 5: A stationary method $t^{(i+1)} = Rt^{(i)} + d$ converges if the spectral radius of its iteration matrix R is less than 1, that is to say if $\rho(R) < 1$..

Definition 5: (Saad (2000)).$A = (a_{ij})$ is called an irreducible matrix if the directed graph associated to A is strongly connected.

Theorem: if A is irreducible matrix with weak diagonal dominance, then the AOR (RPAOR) method converges for any arbitrary choice of the initial $t^{(0)}$ approximation.

Proof: Let t^* be the real solution and let $\bar{t}^{(i+1)}$ be the $(i+1)^{th}$ approximate to the solution (8) by the method (12). Then, we have

$$\left\|\bar{t}^{(i+1)} - t^*\right\| = \left\|\bar{t}^{(i+1)} + \omega(I - rL)^{-1}(b - At^{(i+1)} - t^*\right\|$$

$$\leq \left\|\bar{t}^{(i+1)} - t^*\right\| + |\omega.|\left\|(I - rL)^{-1}\right\|\left\|(b - At^{(i+1)}\right\|$$

We know $\left\|\bar{t}^{(i+1)} - t^*\right\| \to 0$ and $\left\|(b - At^{(i+1)}\right\| \to 0$

Then $\left\|\bar{t}^{(i+1)} - t^*\right\| \to 0$

Hence, the refinement AOR (RPAOR) method converges to the solution of the linear system (8)

Theorem 2: For any initial estimation $t^{(0)}$, the refinement of PAOR method converges to the true solution twice as fast as the PAOR method.

Proof: See Vatti et al. (2018)

Numerical Experiment

The formulation of the RPAOR iteration is validated with the following matrix and can be found in Abdullahi & Ndanusa (2021). The Computation was carried out using Maple 19 software with accuracy of 10 decimal places and the results are presented in table 1-3. From the tables, $\rho(AOR)$ represent the spectral radius of iterative method of classical Accelerated Over Relaxation (AOR), $\rho(PAOR), \rho(RAOR), \rho(RPAOR_1),$ and $\rho(RPAOR_2)$ represent spectral radius of iterative method for Preconditioned (AOR) of Ismaila et al. (2023), spectral radius of Refinement of AOR, spectral radius of iterative method of Ndanusa et al., (2021) and the spectral radius of the New method called Refinement of Preconditioned Accelerated Over Relaxation.

Let consider the coefficient matrix A of the linear system (8) be given by

$$A = \begin{pmatrix} 1 & -1/6 & -1/7 & -1/8 & -1/6 & -1/7 \\ -1/8 & 1 & -1/6 & -1/7 & -1/8 & -1/6 \\ -1/6 & -1/8 & 1 & -1/6 & -1/7 & -1/8 \\ -1/7 & -1/6 & -1/8 & 1 & -1/6 & -1/7 \\ -1/8 & -1/7 & -1/6 & -1/8 & 1 & -1/6 \\ -1/6 & -1/8 & -1/7 & -1/6 & -1/8 & 1 \end{pmatrix}$$

Table 1. Comparison of the spectral radii within AOR of the problem

ω	r	$\rho(AOR)$	$\rho(\text{PAOR}_1)$	$\rho(RAOR)$	$\rho(RPAOR_1)$	$\rho(R\text{PAOR}_2)$
0.95	0.85	0.6205255277	0.4718225724	0.3850519305	0.2323566801	0.2226165407
0.90	0.80	0.6518574112	0.5147480406	0.4249180806	0.2775731577	0.2649655445
0.80	0.70	0.7083014149	0.5917832606	0.5016908907	0.3671890584	0.3502074269
0.70	0.65	0.7516743194	0.6516690078	0.5650142820	0.4426069615	0.4246724970
0.60	0.50	0.8026767336	0.7211254715	0.6442899392	0.5406080078	0.5200219432
0.50	0.40	0.8429614522	0.7769215150	0.7105840063	0.6236885955	0.6036070386
0.40	0.30	0.8796743773	0.8282069826	0.7738270125	0.7041584949	0.6859268053
0.30	0.20	0.9133536720	0.8756791423	0.8342149333	0.7819889721	0.7668139481
0.20	0.10	0.9444202400	0.9198700160	0.8919295900	0.8572013474	0.8461608457
0.10	0.05	0.9727210830	0.9605794594	0.9461863097	0.9285657895	0.9227128962

Table 2. Rates of Convergence of various iterative methods

ω	r	$R(AOR)$	$R(\text{PAOR})$	$R(AOR)$	$R(\text{RPAOR}_1)$	$R(\text{RPAOR}_2)$
0.95	0.85	0.2072403474	0.3262212859	0.4144806949	0.6338448373	0.6524425702
0.90	0.80	0.1858473925	0.2884052979	0.3716947889	0.5566225340	0.5768105970
0.80	0.70	0.1497818907	0.2278373236	0.2995637845	0.4351102685	0.4556746480
0.70	0.65	0.1239702870	0.1859729328	0.2479405743	0.3539817596	0.3719458642
0.60	0.50	0.0954593253	0.1419891641	0.1909186503	0.2671175256	0.2839783302
0.50	0.40	0.0741922848	0.1096228516	0.1483845718	0.2050321972	0.2192457046
0.40	0.30	0.0556780577	0.0818611123	0.1113561141	0.1523295771	0.1637222250
0.30	0.20	0.0393610209	0.0576549946	0.0787220401	0.1067993715	0.1153099960
0.20	0.10	0.0248347141	0.0362735371	0.0496694280	0.0669171550	0.0725470746
0.10	0.05	0.0120116710	0.0174667044	0.0240233400	0.0321873208	0.0349334095

Table 3. Ratio of Rates of Convergence

ω	r	$\dfrac{R(\text{RPAOR}_2)}{R(RAOR)}$	$\dfrac{R(\text{RPAOR}_2)}{R(\text{PAOR})}$	$\dfrac{R(\text{RPAOR}_2)}{R(\text{RPAOR}_1)}$
0.95	0.85	1.5741205277	1.999999995	1.0293411443
0.90	0.80	1.5518393429	2.000000005	1.0362688569
0.80	0.70	1.5211272912	2.000000003	1.0472624550

0.70	0.65	1.5001411743	1.999999993	1.0507486732
0.60	0.50	1.4874310593	2.000000014	1.0631212969
0.50	0.40	1.4775505432	2.000000012	1.0693232946
0.40	0.30	1.4702580662	2.000000006	1.0747894677
0.30	0.20	1.4647739806	2.000000119	1.0796879649
0.20	0.10	1.4605981489	2.000000009	1.0841326805
0.10	0.05	1.4541445715	2.000000044	1.0853158519

Discussion of Results

The corresponding iterative matrices of the matrix A for various iteration processes discussed here are computed alongside with their spectral radii. Five different iterative processes are performed with matrix A; AOR, Preconditioned AOR of Ismaila et al. (2023), Refinement of AOR, Refinement of Preconditioned AOR of Ndanusa et al., (2021) and Refinement of Preconditioned AOR of the new method. The spectral radii of iterative matrices of the various iterations, alongside with varied values of relaxation and accelerated parameters ω and r respectively, are computed and presented in table 1. It reveals that, $\rho(RPAOR_2) < \rho(RPAOR_1) < \rho(RAOR) < \rho(PAOR) < \rho(AOR) < 1.$ which indicate the efficiency of the $RPAOR_2$ iteration over other methods. Also, convergence is observed to be faster when the relaxation and acceleration parameters, ω and r are approaching 1. In other tables, the rate of convergence of the five iterations are compared, where it is shown that in the best case scenario, the $RPAOR_2$ converges thrice as fast as the AOR method, the $RPAOR_2$ converges twice as fast as the PAOR method, $RPAOR_2$ converges one as fast as the $RPAOR_1$ method, and $RPAOR_2$ method converges one and a half times as fast as the RAOR method

Conclusion

It can be clearly observed in this paper that the spectral radius of Refinement Preconditioned Accelerated Over Relaxation AOR (RPAOR) method is smaller compared to spectral radii of the other methods discussed.

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