Sub-Sahara African Academic
Research Publications
March, 2022, Editions

JASOR
Journal of Advanced Science and Optimization Research
Vol. 23, No.9, ISSN 2418-9325

BOUNDARY CHARACTERISTIC ORTHOGONAL POLYNOMIALS VIA GALERKIN WEIGHTED RESIDUAL AND RAYLEIGH-RITZ VARIATIONAL METHOD.

***LIBERTY EBIWAREME; & *EDMOND OBIEM ODOK**
**Department of Mathematics, Rivers State University, Port Harcourt, Nigeria.*
***Department of Mathematics, Cross Rivers University of Technology, Calabar, Nigeria.*

Abstract:
Several problems which feature prominently in mathematics and engineering are modelled in the form of Initial and boundary value problems. In this research article, we explore analytical solution of sample initial and boundary value problem using boundary characteristics orthogonal polynomials through Galerkin and Rayleigh methods. With the of the Gram-Schmidt orthogonalization procedure, the proposed methods were formulated and applied successfully to solve few problems for analytical solution. The absolute errors resulting from both, and methods are compared, and the results presented in tables and figures. The result obtained agrees with the exact solution to certain degrees before diverging. The proposed methods are reliable, efficient, and accurate.

Keyword: Galerkin *Weighted Residual Method (WRM), Rayleigh-Ritz Method, Orthogonal Polynomials, Variational Methods, Orthonormal sets, Orthogonality condition*

Introduction

Boundary value problems denoted BVP are simply linear or nonlinear differential equations where the endpoints are specified within a given interval. These equations find useful applications especially in the mathematical sciences, social sciences, and Engineering. They also feature preeminently in the analysis of beams and structures in Civil Engineering. Many varied methods have been applied to solve these problems prior ranging from analytical, approximate, and semi-analytical methods. Depending on the nature in which the problem is posed, some of the problems doesn't have analytical solutions. Due to this drawback, academics devised ingenious approximate methods to tackling them. Some of these methods include Finite element methods (FEM), Finite Volume methods (FVM), Finite difference methods (FDM) also called Keller-Box method, Runge-Kutta (RK), Euler method, Predictor-Corrector method, and several others.

Due to advances in technology and the advent of powerful Computers, more compact semi-analytical methods have been developed to solve these equations. They include Adomian decomposition method proposed by G. Adomian (1980), Variational Iteration method (VIM) which was the brainchild of Musca, Differential Quadrature method (DQM), Differential transformation method (DTM), Homotopy Analysis method (HAM), Homotopy Perturbation method (HPM) and optimal Homotopy asymptotic method (OHAM). Others are Abkari-Gari method (AGM), Spectral Homotopy analysis method (SHAM), Boundary Characteristics Orthogonal polynomial (BCOP) either with Galerkin or Collocation methods.

In this present study, we apply the weighted residual Galerkin as well as the variational Rayleigh-Ritz method to find analytical solution to boundary value problems. The effectiveness of these methods will be confirmed by comparing the solutions obtained with method to the exact solution and the resulting absolute error will be observed. Results are presented in tables and in figures. The study is organized as follows: The introduction gives detailed detour into boundary value problems and methods advanced thus far to seek for their solution is presented. Section 2 and 3, treats the fundamentals of the Galerkin and Rayleigh-Ritz methods. The similarities between the methods are given in section 4. Numerical examples are solved in section 5, to show the efficiency and accuracy of the proposed methods as well as the resulting errors when compared with the exact solution. In section 6, we draw the conclusion of the study and the discussions of the obtained results. To the best of our knowledge, these methods have been comparatively used to treat boundary problems using the same approach. Therefore, the novelty in this work is the easy convergence of the solved problems with the existing exact solutions.

Gram-Schmidt Orthogonalization Process

The Gram-Schmidt Orthogonalization process is the method of constructing an orthogonal set over an arbitrary closed interval with respect to an arbitrary weight function using non-orthogonal set of linearly independent functions.

Consider an orthogonal polynomial of degree i, given by $\phi_i(x)$, then the sequence of orthogonal polynomials $\{\phi_i(x)\}$, valid in the interval $[a,b]$ with respect to the leading term x^i and weight function, $w(x)$ can be generated using the relation

$$\phi_i(x) = x^i - \sum_{j=0}^{i-1} \alpha_{ij}\phi_j(x), i = 1,2,..,n \quad (1)$$

Where α_{ij} and $\phi_0(x) = 1$ are constants

To obtain α_{ij}, we multiply both sides of Eq. (1) by $w(x)\phi_n(x), 0 \le j \le i-1$ and integrating over the interval $[a,b]$, we obtain

$$\int_a^b \phi_i(x)\phi_n(x)w(x)dx = \int_a^b x^i\phi_n(x)w(x)dx - \int_a^b \sum_{j=0}^{i-1} \alpha_{ij}\phi_j(x)\phi_n(x)w(x)dx \quad (2)$$

Using the orthogonality property of polynomials, we have

$\int_a^b x^i\phi_n(x)w(x)dx - \int_a^b \sum_{j=0}^{i-1} \alpha_{in}{\phi_n}^2(x)w(x)dx = 0$, which yield the constant

$$\alpha_{in} = \frac{\langle x^i,\phi_n(x)\rangle}{\langle \phi_n(x),\phi_n(x)\rangle} = \frac{\int_a^b x^i\phi_n(x)w(x)dx}{\int_a^b \phi_n^2(x)w(x)dx}, 0 \le n \le i-1 \quad (3)$$

Putting $i = 1$ in Eq. (1), the first linear polynomial with leading term, x is obtained using the relation

$$\phi_1(x) = x + \alpha_{1,0}\phi_0(x)$$

where $\alpha_{1,0} = -\frac{\langle x,,\phi_0(x)\rangle}{\langle \phi_0(x),\phi_0(x)\rangle} = -\frac{\int_a^b x\phi_0(x)w(x)dx}{\int_a^b \phi_0^2(x)w(x)dx}$ (4)

Similarly, the next polynomial of degree two, with leading term x^2 is obtained using the relation

$$\phi_2(x) = x^2 + \alpha_{2,0}\phi_0(x) + \alpha_{2,1}\phi_1(x)$$

The constants $\alpha_{2,0}$ and $\alpha_{2,1}$ are obtained using the orthogonality condition as follows

Now, $\phi_2(x)$ and $\phi_0(x)$ are orthogonal, so we have

$$\int_a^b w(x)\phi_0(x)\left[x^2 + \alpha_{2,0}\phi_0(x) + \alpha_{2,1}\phi_1(x)\right]dx \tag{5}$$

Simplifying, the constant become

$$\alpha_{2,0} = -\frac{\langle x^2,,\phi_0(x)\rangle}{\langle \phi_0(x),\phi_0(x)\rangle} = -\frac{\int_a^b x^2\phi_0(x)w(x)dx}{\int_a^b \phi_0^2(x)w(x)dx} \tag{6}$$

Similarly, $\phi_2(x)$ and $\phi_1(x)$ are also orthogonal, hence using the relation

$$\int_a^b w(x)\phi_1(x)\left[x^2 + \alpha_{2,0}\phi_0(x) + \alpha_{2,1}\phi_1(x)\right]dx = 0 \tag{7}$$

Using the orthogonality condition and simplifying we get the second constant as

$$\alpha_{2,0} = -\frac{\langle x^2,,\phi_1(x)\rangle}{\langle \phi_1(x),\phi_1(x)\rangle} = -\frac{\int_a^b x^2\phi_1(x)w(x)dx}{\int_a^b \phi_1^2(x)w(x)dx} \tag{8}$$

Plugging Eqs. (6) and (8) into $\phi_2(x)$, we get the second-degree orthogonal polynomial in x

Proceeding in the same manner, the orthogonal polynomials are given in general as

$\phi_j(x) = x^j + \alpha_{j,i}\phi_0(x) + \alpha_{j,1}\phi_1(x) + \cdots + \alpha_{j,j-1}\phi_{j-1}(x),$ where the constants are so chosen such that$\alpha_j(x)$ are orthogonal to the set, $\phi_0(x), \phi_1(x), \phi_2(x), \ldots .\phi_{j-1}(x)$ and yield the constants as

$$\alpha_{ij} = -\frac{\langle x^j,,\phi_i(x)\rangle}{\langle \phi_i(x),\phi_i(x)\rangle} = -\frac{\int_a^b x^j\phi_i(x)w(x)dx}{\int_a^b \phi_i^2(x)w(x)dx} \tag{9}$$

Generation of BCOPs.

In the generation of the boundary characteristics orthogonal polynomials, the simplest polynomial of the least order which satisfies the boundary conditions is taken as the first member of the BCOPs. Subsequent members of the orthogonal set in the interval $[a, b]$ are generated using the Gram-Schmidt procedure as follows.

$$\phi_1(x) = (x - \alpha_1)\phi_0(x)$$
$$\phi_2(x) = (x - \alpha_2)\phi_1(x) - \beta_2\phi_0(x)$$
$$\vdots$$
$$\phi_k(x) = (x - \alpha_k)\phi_{k-1}(x) - \beta_k\phi_{k-2}(x) \qquad (10)$$

where, $\alpha_k = \frac{\int_a^b x\phi_{k-1}^2(x)dx}{\int_a^b \phi_{k-1}^2(x)dx}$ and

$$\beta_k = \frac{\int_a^b x\phi_{k-1}(x)\phi_{k-2}(x)dx}{\int_a^b \phi_{k-2}^2(x)dx}$$

In the above, we have taken the weight function, $w(x) = 1$ and $\phi_k(x)$ satisfy the orthogonality condition, $\int_a^b \phi_i(x)\phi_j(x)dx = \delta_{ij}$

Fundamentals of the Galerkin Weighted Residual Method with BCOPs

Consider a second order boundary value problem given by

$$y'' + p(x)y = q(x), y(a) = \alpha_1, y(b) = \alpha_2 \qquad (11)$$

Let the approximate solution of degree three be considered as

$$y(x) \approx \hat{y}(x) = \sum_{i=0}^{n} c_i\phi_i(x) \qquad (12)$$

where $\phi_i(x)$ are the boundary characteristics orthogonal polynomials that satisfies the given boundary conditions and $c_0, c_1, \dots, c_n$ are arbitrary constants. Plugging Eq. (12) into (11) give rise to a non-zero residual, R of the form

$$R(x; c_0, c_1, \dots, c_n) = \sum_{i=0}^{n} c_i\phi_i''(x) + p(x)\sum_{i=0}^{n} c_i\phi_i(x) - q(x) \qquad (13)$$

Next for each of the $(n + 1)$ $BCOPs$, $\phi_0(x), \phi_1(x), \dots, \phi_n(x)$, we orthogonalize the residual, R as follows

$$\int_a^b R(x; c_0, c_1, \dots, c_n)\phi_j(x)dx = 0, j = 0,1,2, \dots, n \qquad (14)$$

where $\int_a^b \phi_i(x)\phi_j(x)dx = 0, i \neq j$

Eq. (14) generate an $(n + 1)$ system of linear equations with $(n + 1)$ unknowns. Solving these resulting systems gives the constants, $c_0, c_1, \dots, c_n$. Substituting these constants into Eq. (12) gives the approximate solution for the given BVP. This method is efficient in that the terms containing the linearly independent set (BCOPs), $\phi_0(x), \phi_1(x), \ldots, \phi_n(x)$ all vanish in view of the orthogonality property.

Basics of the Rayleigh-Ritz Variational Method with BCOPs

Consider a second-order boundary value problem defined as follows

$$\frac{d^2y}{dx^2} + p(x)y + q(x) = 0 \quad (15)$$

Subject to the boundary conditions

$$y(a) = y(b) = 0 \quad (16)$$

Let $u(x) = \sum_{i=1}^{n} \alpha_i \phi_i$ be an approximate solution of Eq. (15), where ϕ_i are known functions usually polynomials, which are linearly independent and satisfy the essential boundary conditions.

$$\phi_i(a) = \phi_i(b) = 0 \quad (17)$$

The functional for Eq. (15) is given by

$$I[u(x)] = \int_a^b \left[\left(\frac{du}{dx} \right)^2 - pu(x)^2 - 2qu(x) \right] dx \quad (18)$$

Substituting for $u(x)$ in Eq. (18), the integral $I\big(u(x)\big)$ will be a minimum

That is, $I[u(x)] = \int_a^b f\big(x, u(x), u'(x)\big) dx =$ Minimum

$$I(\alpha_1, \alpha_2, \ldots, \alpha_n) = \int_a^b \left[\left(\frac{d}{dx} \sum_{i=1}^{n} \alpha_i \phi_i \right)^2 - p\left(\sum_{i=1}^{n} \alpha_i \phi_i\right)^2 - 2q \sum_{i=1}^{n} \alpha_i \phi_i \right] dx \quad (19)$$

The necessary condition for the existence of extremum is given by

$$\frac{\partial I[u(x)]}{\partial \alpha_i} = \int_a^b \left[\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial \alpha_i} + \frac{\partial f}{\partial u'} \cdot \frac{\partial u'}{\partial \alpha_i} \right] dx$$

$$\frac{\partial I[u(x)]}{\partial \alpha_i} = \int_a^b \left[\frac{\partial f}{\partial u} \alpha_i + \frac{\partial f}{\partial u'} \alpha_i' \right] dx = 0, i = 1,2, \ldots n \quad (20)$$

Writing the above in explicit form, we have

$$\frac{\partial I}{\partial \alpha_1} \delta\alpha_1 + \frac{\partial I}{\partial \alpha_2} \delta\alpha_2 + \cdots + \frac{\partial I}{\partial \alpha_n} \delta\alpha_n = 0 \quad (21)$$

Since the $\delta\alpha_i$ are arbitrary, Eq. (21) is written as

$$\frac{\partial I}{\partial \alpha_i} = 0, i = 1,2,\dots,n \qquad (22)$$

Eq. (22) gives a system of equations with unknown α_i, solving the resulting system for the unknown functions and substituting into the approximate solution gives the required solution.
The Rayleigh-Ritz method converges to the actual solution provided that the functions, $\phi_i(x)$ are linearly independent and satisfy the essential boundary condition.

Computational Results and Analysis

In this section, we consider examples of three boundary value problems to demonstrate the efficiency, accuracy, and simplicity of the proposed methods. The absolute error obtained from both methods are also examined to know their degree of convergence.
Example 6.1 Solve the boundary value problem

$$y'' - y + x = 0, \ (0 \le x \le 1) \qquad (23)$$

Subject to the boundary condition

$$y(0) = y(1) = 0 \qquad (24)$$

Solution via Rayleigh-Ritz Variational Method

The functional for the given differential equation is given by

$$I\big(y(x)\big) = \int_a^b \left[\left(\frac{dy}{dx}\right)^2 - p(x)y - 2q(x)y\right] dx \qquad (25)$$

$$I\big(y(x)\big) = \int_a^b (2xy - y^2 - (y')^2) dx \qquad (26)$$

Where $f(x, y, y') = 2xy - y^2 - (y')^2$
The condition for $f(x, y, y')$ to have an extremum is that it satisfies the Euler-Lagrange equation,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Now, assume a trial solution that satisfies the boundary solution of the form

$$\tilde{y}(x) = c_0 + \sum_{i=1}^{3} c_i \phi_i \quad (27)$$

Applying the boundary conditions

$$y(0) = 0, \Longrightarrow c_0 = 0$$
$$y(1) = 0, \Longrightarrow c_1 + c_2 + c_3 = 0$$
$$c_1 = -c_2 - c_3$$

The approximate solution now takes the form

$$\tilde{y}(x) = (x^2 - x)c_2 + (x^3 - x)c_3 \quad (28)$$

Substituting Eq. (27) into Eq. (26), we have the equivalent form as

$$I\big(y(x)\big) = \int_a^b \left(2x\tilde{y}(x) - \tilde{y}(x)^2 - \left(\tilde{y}'(x)\right)^2\right) dx \quad (29)$$

$$I(y) = -\frac{1}{6}c_2 - \frac{4}{15}c_3 - \frac{11}{10}c_2^2 - \frac{92}{105}c_3^2 - \frac{11}{10}c_2 c_3 \quad (30)$$

The stationary values of Eq. (29) are obtained when $\frac{\partial I(y)}{\partial c_2} = \frac{\partial I(y)}{\partial c_3} = 0$, the resulting equations become

$$\frac{11}{5}c_2 + \frac{11}{10}c_3 = -\frac{1}{6} \quad (31)$$

$$\frac{11}{10}c_2 + \frac{184}{105}c_3 = -\frac{4}{15} \quad (32)$$

Solving Eqs. (30) & (31), we have

$c_2 = 0.162014, c_3 = -0.475543$

The approximate now become

$\tilde{y}(x) = 0.313529x + 0.162014x^2 - 0.475543x^3$

Solution by Galerkin Method

Recall the trial solution in Eq. (27), we have

$$\tilde{y}(x) = (x^2 - x)c_2 + (x^3 - x)c_3$$

Differentiating the above twice and substituting into the given equation, the residual become

$$R = (x - x^2 - 2)c_2 + (7x - x^3)c_3 + x \neq 0 \quad (33)$$

The inner product of the weighted function and residual error becomes
$\langle w_i, R\rangle = 0,$

$$\int_0^1 w_i R dx = 0 \quad (34)$$

For $i = 1, w_1 = x^2 - x$

$\int_0^1 (x^2 - x)[(x - x^2 - 2)c_2 + (7x - x^3)c_3 + x]dx = 0$ reduce to

$$18c_2 - 33c_3 = 5 \quad (35)$$

Similarly, for $i = 2, w_2 = x^3 - x$, we obtain

$$\int_0^1 (x^3 - x)[(x - x^2 - 2)c_2 + (7x - x^3)c_3 + x]dx = 0$$

$$47.25c_2 - 92c_3 = 14 \quad (36)$$

Solving Eqs. (34) and (35), we obtain the results for the unknowns

$$c_2 = -0.0206718, c_3 = -0.162791$$

The approximate solution via Galerkin method is

$$\tilde{y} = 0.183463x - 0.0206718x^2 - 0.162791x^3 \quad (37)$$

Example 6.2 Solve $\frac{d^2y}{dx^2} = 3x - 4y$, subject to the condition, $y(0) = 1, y(1) = 1$

Implementation using Galerkin Method

$$\frac{d^2y}{dx^2} + 4y - 3x = 0,\ y(0) = 0, y(1) = 1 \quad (38)$$

Let the approximate solution of the given equation be

$$\hat{y} = \hat{y} = c_0 + c_1x + c_2x^2 + c_3x^3 \quad (39)$$

Putting the boundary condition$y(0) = 0$ and $y(1) = 1$ gives $c_0 = 0, c_1 = 1 - c_2 - c_3$

The approximate solution is now of the form

$$\hat{y} = x + (x^2 - x)c_2x + (x^3 - x)c_3 \quad (40)$$

Differentiating Eq. (39) $w.r.t.x$, twice and substituting into eq. (37) we get the residual error as

$$R = (4x^2 - 4x + 2)c_2 + (4x^3 + 2x)c_3 + x \neq 0 \tag{41}$$

Hence the inner product of the weighted function and residual error becomes,

$\langle w_{i,}, R\rangle = 0$

$\int_0^1 w_i R dx = 0$

For $i = 1,\ w_1 = x^2 - x$

$\int_0^1 (x^2 - x)[(4x^2 - 4x + 2)c_2 + (4x^3 + 2x)c_3 + x]dx = 0$ reduced to

$$12c_2 + 18c_3 = -5 \tag{42}$$

Similarly, for $i = 2, w_2 = x^3 - x$ we obtain

$$\int_0^1 (x^3 - x)[(4x^2 - 4x + 2)c_2 + (4x^3 + 2x)c_3 + x]dx = 0$$

$$-\frac{3}{10}c_2 - \frac{52}{105}c_3 = \frac{2}{15} \tag{43}$$

Solving Eqs. (41)−(42), we get the constants

$c_2 = -0.141224$ and $c_3 = 0.183628$

The approximate solution is given by

$$\hat{y} = 0.9576x - 0.141224x^2 + 0.183628x^3 \tag{44}$$

Implementation via Rayleigh-Ritz Method

The functional of the equation is given by

$$I(y) = \int_0^1 (-6xy + 4y^2 - (y')^2)dx \tag{45}$$

Where $f(x, y, y') = -6xy + 4y^2 - (y')^2$

The function, $f(x, y, y')$ have extremum if it satisfies the Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

Assume an approximate solution of Eq. (37) of the form

$\tilde{y}(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ subject to the boundary condition

$$\tilde{y}(0) = 0, \tilde{y}(1) = 1 \tag{46}$$

Applying the boundary conditions, $y(0) = 0$ and $y(1) = 1$, we have the constants as

$c_0 = 0$ and $c_1 = -c_2 - c_3$
Substituting the constants into the approximate solution give

$$\tilde{y}(x) = (x^2 - x)c_2 + (x^3 - x)c_3 \quad (47)$$

The functional in Eq. (45) now takes the form

$$I(y) = \int_0^1 (-6x\tilde{y} + 4\tilde{y}^2 - (\tilde{y}')^2)dx \quad (48)$$

Plugging Eq. (47) into Eq. (48), we have the equivalent form

$$I(y) = \frac{c_2}{2} + \frac{7c_2^2}{15} + \frac{4c_3}{5} + \frac{7c_2c_3}{5} + \frac{116c_3^2}{105} \quad (49)$$

Differentiating Eq. (49) partially $w.r.t$ c_2 and c_3 and applying the stationary condition give the systems of equations

$$28c_2 + 42c_3 = -15 \quad (50)$$

$$147c_2 + 232c_3 = -84 \quad (51)$$

Solving the systems in Eqs. (50) and (51) give the constants

$$c_2 = 0.149068, c_3 = -0.456522$$

The approximate solution now becomes

$$\tilde{y}(x) = 0.307454x + 0.149068x^2 - 0.456522x^3$$

Example 6.3 Consider the boundary value problem

$$-y'' + xy = 0 \quad (52)$$

Subject to the boundary condition

$$y(0) + y'(0) = 1, y(1) = 1 \quad (53)$$

Solution using Rayleigh-Ritz Method

The functional of the Eq. (52) is given by

$$I(y) = \frac{1}{2}\int_0^1 [(y')^2 + xy^2]dx + \frac{1}{2}[2y(0) - y_0^2] \quad (54)$$

Let a two-point approximation of the problem be given by

$$\tilde{y}(x) = 1 + (1 - x)(c_1 + c_2x) \quad (55)$$

Plugging Eq. (55) into the functional in Eq. (52) and integrating takes the form

$$I(y) = \frac{1}{2}\int_0^1 [(\tilde{y}')^2 + x\tilde{y}^2]dx + \frac{1}{2}[2y(0) - y_0^2]$$

$$I(y) = \frac{1}{2}\left[\frac{1}{12}c_1^2 + \frac{7}{20}c_2^2 + \frac{1}{15}c_1c_2 + \frac{1}{3}c_1 + \frac{1}{6}c_2 + \frac{3}{2}\right] \quad (56)$$

The necessary condition for the existence of an extremum gives the equation

$\frac{\partial I(y)}{\partial c_1} = \frac{1}{2}\left[\frac{1}{6}c_1 + \frac{1}{15}c_2 + \frac{1}{2}\right] = 0$ or

$$5c_1 + 2c_2 = -10 \quad (57)$$

$\frac{\partial I(y)}{\partial c_2} = \frac{1}{2}\left[\frac{1}{15}c_1 + \frac{7}{10}c_2 + \frac{1}{6}\right] = 0$ which reduced by

$$2c_1 + 21c_2 = -5 \quad (58)$$

The solutions of the systems in Eqs. (57) and (58) gives

$$c_1 = -1.980198, c_2 = -0.049505$$

Hence the approximate solution of the given problem become

$$\tilde{y}(x) = 1 - (1 - x)(1.900198 + 0.049505x) \quad (59)$$

Solution using Galerkin method

Let the approximating function be a third-degree polynomial of the form

$$\tilde{y}(x) = c_0 + c_1x + c_2x^2 + c_3x^3 \quad (60)$$

Applying the boundary conditions

$y(0) + y'(0) = 1 \Longrightarrow c_0 + c_1 = 1$

$$y(1) = 1, \Longrightarrow c_2 = -c_3$$

$$\tilde{y}(x) = 1 + (x - 1)c_1 + (x^3 - x^2)c_3 \quad (61)$$

Differentiating Eq. (61) twice and substituting into Eq. (52), we have the residual error as

$$R = (x - x^2)c_1 + (x^2 - x^4 + 6x)c_3 - x \neq 0$$

For $i = 1, w_1 = x - 1$, then the inner product become

$$\int_0^1 w_1 R dx = 0$$

$$\int_0^1 (x-1)[(x-x^2)c_1 + (x^2 - x^4 + 6x)c_3 - x]dx = 0$$

$$\frac{11}{12}c_1 - \frac{21}{20}c_3 = \frac{1}{6} \quad (62)$$

Similarly, for $i = 2, w_2 = x^3 - x$, we have the integral form as

$$\int_0^1 (x^3 - x)[(x-x^2)c_1 + (x^2 - x^4 + 6x)c_3 - x]dx = 0$$

$$\frac{1}{20}c_1 + \frac{101}{120}c_3 = \frac{2}{15} \quad (63)$$

Solving the systems in Eqs. (62) and (63), we obtain the solutions as

$$c_1 = 0.340131, c_2 = 0.13821$$

Putting the solutions into the approximating function in Eq. (61), we have

$$\tilde{y}(x) = 0.659869 + 0.201921x + 0.13821x^3 \quad (64)$$

Table 1. Comparison between Exact solution, $\hat{y}(x)$ and the approximate solutions using Galerkin and Rayleigh-Ritz for Example 6.1 (Ebiwareme and Miracle, 2020)

t	**Exact Solution**	**Galerkin Solution**	**Rayleigh Solution**	**Absolute Error for Galerkin**	**Absolute Error for Rayleigh**
0	0.0000	0.00000	0.0000	0.00000	0.0000
1	0.0000	2.0000E-7	0.00000	2.000E-7	0.0000
2	-1.08616	-1.01809	-2.52923	0.068072	1.44307
3	-5.52439	-4.03101	-10.4409	1.49338	4.91656
4	-19.2215	-10.0153	-26.5884	9.20596	7.36693
5	-58.1409	-19.9484	-53.8249	38.1925	4.31598
6	165.6410	-34.8063	-95.0036	130.8350	70.6378
7	-459.572	-55.5660	-152.978	404.006	306.594
8	-1260.28	-83.2043	-230.601	1177.07	1029.67
9	-3438.53	-118.698	-330.726	3319.83	3107.80
10	-9361.36	-163.024	-456.206	9198.340	8905.150

Table 2. Comparison between Exact solution, $\hat{y}(x)$ and the approximate solutions using Galerkin and Rayleigh-Ritz for Example 6.2 (Ebiwareme and Miracle, 2020)

t	Exact Solution	Galerkin Solution	Rayleigh Solution	Absolute Error for Galerkin	Absolute Error for Rayleigh
0	1.0000	0.0000	0.0000	1.00000	1.0000
1	1.0000	1.00000	0.00000	4.000E-6	1.0000
2	0.291927	2.81933	2.4410-	2.52740	2.73293
3	3.00547	6.55974	-10.0621	3.55427	13.0676
4	3.57930	13.3230	-25.6025	9.74371	29.1813
5	2.51238	24.2109	-51.8013	21.6985	54.3137
6	4.95076	40.3252	-91.3975	35.3744	96.3483
7	6.11245	62.7676	-147.131	56.6552	153.243
8	4.83142	92.640	-221.739	87.8086	226.570
9	6.86015	131.044	-317.963	124.184	324.823
10	8.5769	179.082	-438.541	170.505	447.118

Table 3. Comparison between Exact solution, $\hat{y}(x)$ and the approximate solutions using Galerkin and Rayleigh-Ritz for Example 6.3 (Ebiwareme and Miracle, 2020)

t	Exact Solution	Galerkin Solution	Rayleigh Solution	Absolute Error for Galerkin	Absolute Error for Rayleigh
0	-0.981363	0.659869	-0.900198	1.64123	0.081165
1	1.00000	1.00000	1.000000	0.00000	0.000000
2	4.474860	2.16930	2.99921	2.30556	1.475650
3	19.7859	4.99730	5.09743	14.7886	14.6885
4	118.3850	10.3130	7.29465	108.072	111.090
5	928.781	18.9457	9.59089	909.835	919.190
6	9229.26	31.7248	11.9861	9197.54	9217.27
7	113420	49.4793	14.4804	113371.0	113406
8	1.69378E6	73.0388	17.0737	1.6941E6	1.6938E6
9	3.0319E7	103.232	19.7659	3.03189E7	3.0319E7
10	6.433E8	140.889	22.5572	6.433E8	6.433E8

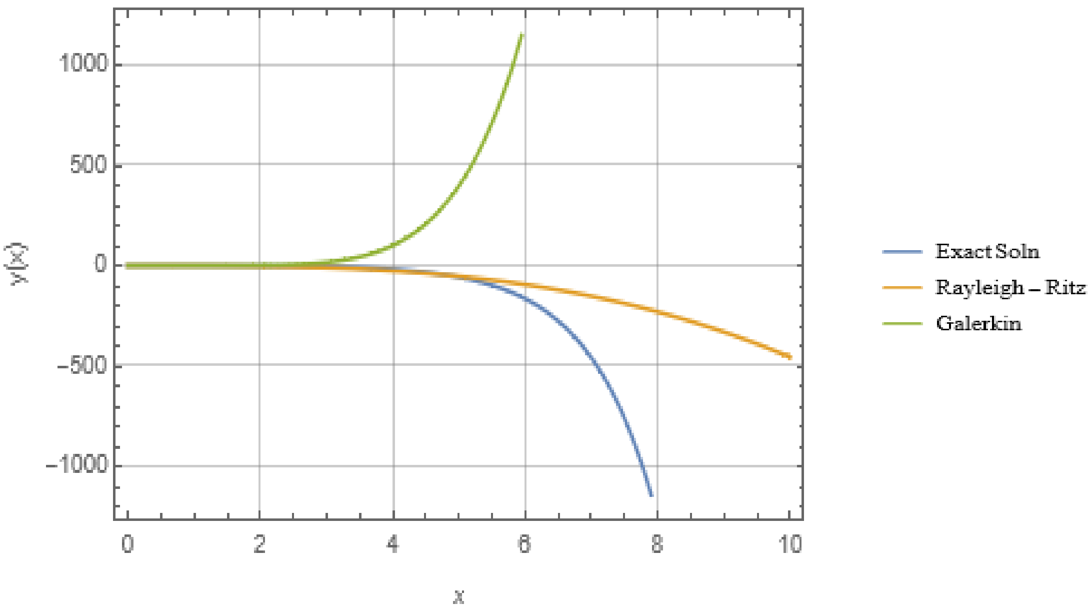

Figure 1. Gives the Exact solution, $\hat{y}(x)$ and the approximate solutions using Galerkin and Rayleigh-Ritz for Example 6.1 (Kaliakin, 2002)

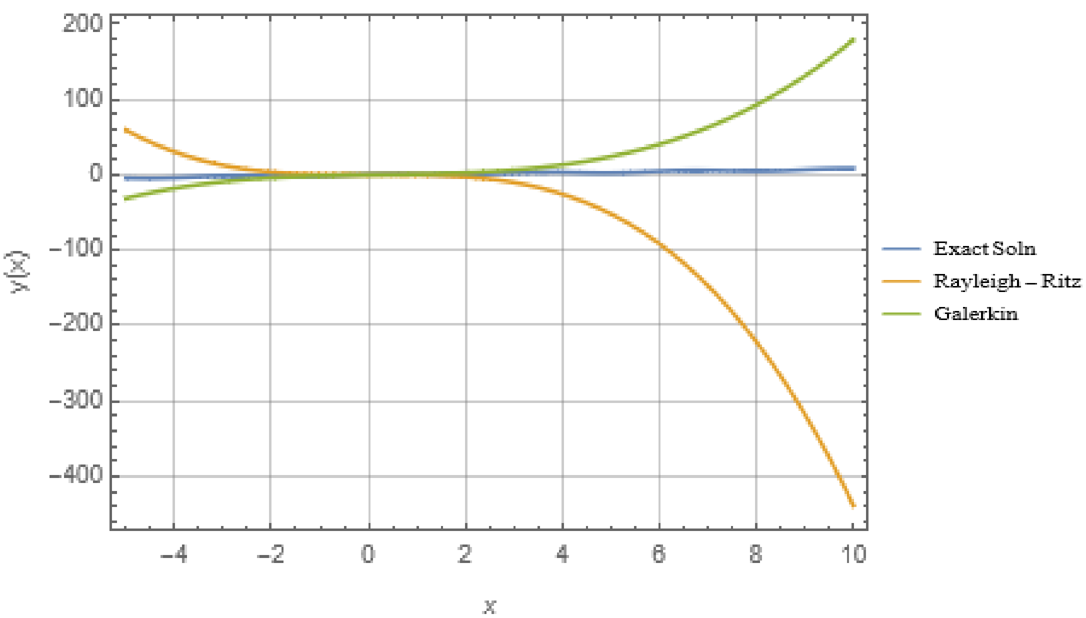

Figure 2. Gives the Exact solution, $\hat{y}(x)$ and the approximate solutions using Galerkin and Rayleigh-Ritz for Example 6.2 (Kaliakin, 2002)

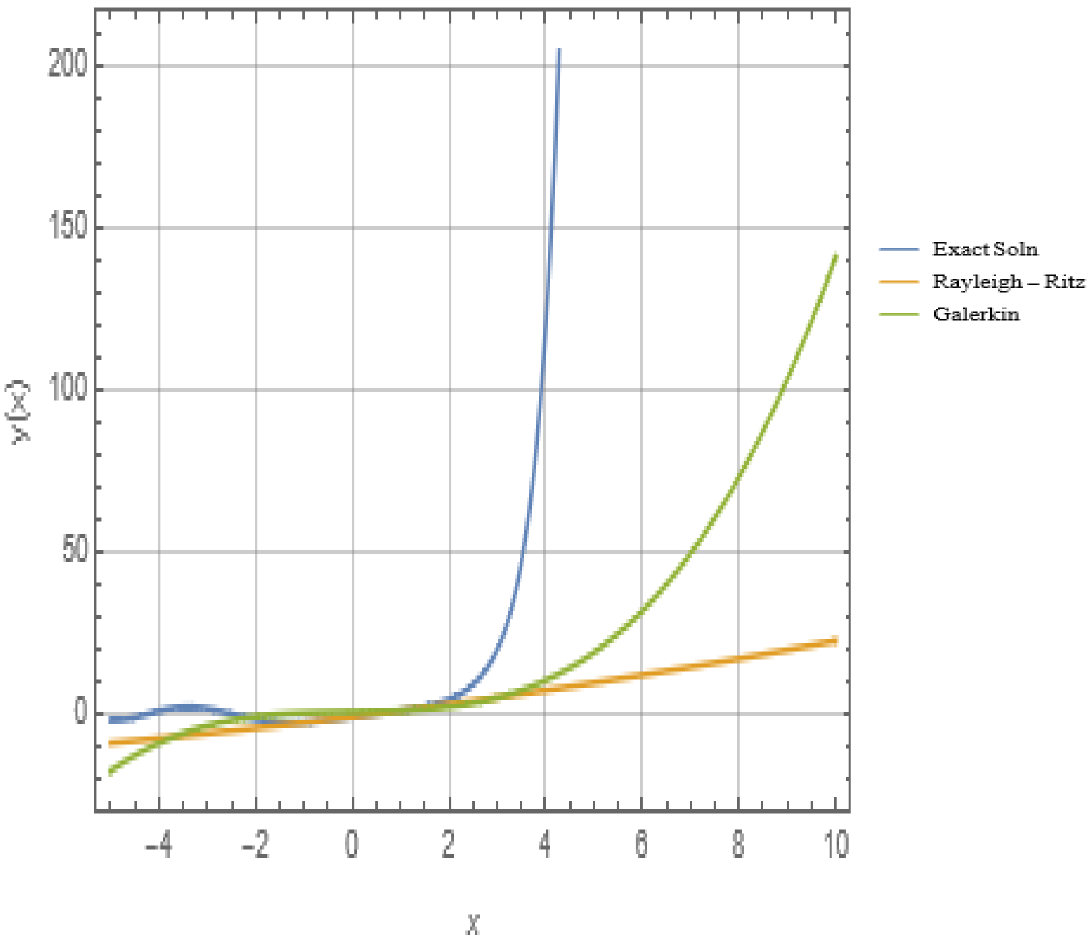

Figure 3. Gives the Exact solution, $\hat{y}(x)$ and the approximate solutions using Galerkin and Rayleigh-Ritz for Example 6.3 (Cecelia, 2014)

Discussion of Results

In this subsection, we present the results obtained using Galerkin and Rayleigh-Ritz variational methods and compare these results with the exact solution. They are presented in tables and in figures. It was observed that the Galerkin method produces a solution which best approximates the exact solution. However, the accuracy of the obtained when more terms are considered. Similarly, the Rayleigh-Ritz gives a better approximate solution compared to the Galerkin method, the only drawback is that it depends on a functional which may or may not be available for all boundary value problems. Nevertheless, for BVP which possess functional, the approximate solution converges to the exact solution with more terms considered.

References.

Baluch, M.H., Mohsen, M.F.N., and Ali, A.I. (1983). Method of Weighted Residuals as Applied to Nonlinear differential equation. Applied Mathematical Modelling, 7(5), 362-365.

Boyd, J.P. (1999). *Chebyshev and Fourier Spectral Method*. Dover Publications, USA.

Cicelia, J. (2013). *Mathematical Methods in Thermal Engineering*, S&S Publication, India.

Cicelia, J. (2014). Solution of Weighted Residual Methods by using the Galerkin Method. Indian Journal of Science and Technology, 17(35), 52-54.

Desai, Y.M., Eldho, T.I., and Shah, A.H. (2011). Finite Element Method with Applications in Engineering. Pearson Educational Inc, Delhi.

Ebiwareme, L., Godwin, M. (2020). The comparison of weighted residual methods of solving boundary value problems in science and Engineering. African Scholars Journal of Pure and Applied Mathematics. Vol. 18, No. 9, ISSN: 2278-1897.

Finlayson, B.A (2013). The Method of Weighted Residual Methods and Variational Principles, Vol. 73, Philadelphia, SIAM.

Franklin, J. (2010). *Advanced mechanics and General Relativity*. Cambridge University Press.

Gerald, C.F. and Wheatley, P.O. (2014). *Applied Numerical Analysis*, seventh edition. Pearson Educational Inc, New York.

Grandin, H. (1991). *Fundamentals of the Finite Element Methods*. Waveland Press, USA.

Hatami, M. (2017). *Weighted Residual Methods: Principles, Modifications and Applications*. Academic Press, London, 2017

Kaliakin, V.N. (2002). *Introduction to Approximate Solution Techniques, Numerical Modelling and Finite Element Methods*. Marcel Decker Inc, New York.

Larry, J. (1982). The Galerkin and Sub domain forms of the weighted residual method are used to generate recursive equation in time for the numerical solution of a system of ordinary differential equations.

Lindgren, L.E. (2009). *From Weighted Residual methods to Finite Element methods*. Technical Report.

Locker, J. (1971). The Method of Least squares for boundary value problems. Transaction of the American Mathematical Society, 154: 57-68.

Logan, D.L. (2011). *A First Course in the Finite Element Methods*, fifth edition. Cengage learning, Stamford, UK.

Mills, P.I. (1982). Investigation of the method of weighted residual applied provide approximate solution to mixed (split) bounding value problem which are described by triple-server relations.

Odejide, S.A. and Aregbesola, Y.A.S. (2011). Application of Method of Weighted Residual to Problems with Semi-infinite Domain. Romanian Journal of Physics, 56(1-2) 14–24.

Salih (2016) study weighted residual method is a generic class of method developed to obtain approximate solution to the differential equation weighted residual method involves two major steps. 2016.

Sheng-wei. C. (2013). Study of collocation method such as he is reproducing kernel collocation method (RKCM), addressed the domain integration issue in the Galerkin type weigh free method, such as the reproducing kernel practice method.

Whu. (2007). Introduces the weighted radial basis collocation method for boundary value problems. We first show that the employment of least-square functional with quadrative rules constitutes an approximation of the direct collocation method.

Zienkiewicz, O.C., and Taylor, R.L. (2000). *Finite Element Methods*, Vol. 1