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INVESTIGATING THE OPTIMALITY CRITERIA FOR PARTIALLY BALANCED LATTICE DESIGNS WITH THREE ASSOCIATE CLASSES

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ABSTRACT

Partially Balanced Lattice Designs (PBLD) is a subclass of Incomplete Block Designs which are similar to the Balanced Lattices only that they allows for more flexible choice of the number of replications, the PBLD requires that the number of treatments must be a perfect square and that Block size k must be equal to the square root of these treatments number. Balanced Lattice Design require the number of replications to be $k+1$. *They exist for certain parameters. They require large number of replications, which consumes logistics, time and effectiveness. The aim of this study was to investigate the Optimality Criteria for a Partially Balanced Lattice Designs with three associate classes. The A-, D-, and G- optimality criteria were studied. These approaches demonstrated in our study involving sixty-four treatments. The results show that D-optimality has the highest values of* $4.2950e+009$ *while A- and G- Optimality have* 4 *and* 16 *respectively. It means that D- criteria is more optimal than A- and G- criteria. In the same manner, the efficiencies of this Design were considered by maximizing the information matrix; the results are* $6.25\%, 3.00\%$ *and* 25% *for A-, D-, and G-efficiency respectively, D-efficiency Criteria also show better result when compared to A- and G-efficiency. Hence, the results indicate that Partially Balanced Lattice Designs with four replications possess D-Optimality Criteria. It is therefore recommended here that for studies in Partial Lattice Designs, D- Optimality is*

better when a researcher is interested in square Lattice Designs. Agriculture researchers, sample surveyors and plant breeders should use Partial Lattice Designs to test a large number of entries that are compare directly for selection, it is cost effective in experimental Designs and improve efficiency. It also serves as reference material for researchers who wishes to carryout research on Partial Lattices.

***Keywords**: optimality criteria, efficiency criteria, association scheme, replication and lattices,*

INTRODUCTION

When the number of treatments is very large and blocking is necessary, Incomplete Block Designs are generally used. The origins of Incomplete Block Design dates back to (Yates, 1936), who introduced the concept of Balanced Incomplete Block Designs and their analysis utilizing both intra- and inter-block information. He referred to these designs as quasi-factorial or Lattice Designs. In order to eliminate heterogeneity; a concept of Balanced Incomplete Block Design was introduced.

The importance of BIB Designs in statistical design of experiments for varietal trials was realized in 1936 when Yates discussed these designs in the context of biological experiments. Yates, (1936) introduced these designs in his paper, "A new method of arranging varietal trials, involving a large number of varieties". Balanced Incomplete Block Designs have several advantages. They are connected designs and the block sizes are equal. A design where all the element contrasts are estimable is a connected design. Otherwise, it is a disconnected Design. Another important property of the BIBD is that it is balanced. This means that all the treatments difference is estimated with the same accuracy. A restriction in using the BIBD is that they are not available for all parameter combinations. They exist only for certain parameters. Sometimes, they require large number of replications and this hampers the utility of the BIBD **(**Rao 1961**).**

Partially Balanced Incomplete Block Design (PBIBD) compromise on this property up to some extent and help in reducing the number of replications. In simple words, the pairs of treatments can be arranged in different sets such that

difference between the treatment effects of a pair and for all pairs in a set is estimated with the same accuracy.

Partially Balanced Incomplete Block Designs remain connected like BIBD but are no more balanced, rather they are Partially Balanced in the sense that some pair of treatments have the same efficiency whereas other pairs have similar efficiency but different from the efficiency of the earlier pairs of treatments. The notion of Partially Balanced Incomplete Block Designs was introduced by Bose and Nair (1939); which encompass some of the Lattice Designs introduced earlier by Yates. There are three main concepts of balancing in Incomplete Block Designs, namely:

i. Variance Balanced
ii. Efficiency Balanced
iii. Neighbor Balanced.

A Block Design with incidence matrix having all elements equal to unity is a randomized (complete) Block Design. It can be verify that such a design is necessarily "orthogonal" also "Variance Balanced". Awad and Barnejee (2013) gave a necessary and sufficient condition for a general block design to be variance balanced. They noted that a block design is said to be balanced if every normalized estimable linear function of treatment effects is estimated with the same variance.

Cervenka and Graczyk (2009) postulated that a block design is balanced if every contrast of treatment effects estimated through the design with the same efficiency factor.

Lattice Design is a method of constructing certain types of resolvable Incomplete Block Designs, some of which are Balanced Incomplete Block (BIB) Designs or Partially Balanced Incomplete Block (PBIB) Designs. Historically, Lattice Design were developed for large-scale agricultural experiments (Yates, 1936) in which large number of varieties are to be compared. Consequently, since then application has been and continues to be in Agricultural experiments.

A special feature of Lattice designs is that the number of treatments "v" is related to the block size "k" in the form $v = k^2$ or $v = k^3$ or $v = k(k+1)$ or $v = {k^2}/{2}$. Even though this limits the number of possible designs, Lattice

Design represent an important class of designs. In certain type of Agronomic or Breeding experiments the number of treatments may be, say, 100 or more but the existing Lattice Designs is classified according to number of treatments, block size and number of restrictions imposed on randomization, it can be broadly classified as square, circular, cubic and rectangular Lattice designs. These designs were discussed in detail in Alabi (2018).

Lattice Designs are one class of Incomplete Block Design most commonly used in Agricultural research. There is sufficient flexibility in the design to make its application simpler than most other incomplete block designs. There are several types of Lattice designs and the two of the most commonly used Lattice Designs are, the Balanced Lattices and the Partially Balanced Lattice Designs. The following basic features characterize the Balanced Lattice design:

- The number of treatments (t) must be a perfect square ($t = k^2$), although this requirement may seem stringent at first; it is usually easy to satisfy in practices. As the number of treatments becomes large, adding a few more or eliminating some less important treatments is usually easy to accomplish. For example, if a plant breeder wishes to test the performance of 80 varieties in a Balanced Lattice Design, all he needs to do is to add one more variety for a perfect square or if he has $82, 83\ or\ 84$ varieties to start he can easily eliminate one, two or three less important treatments in that order.
- The block size (k) is equal to the square root of the number of treatments ($k = \sqrt{v}$).
- The number of replication (r) is one more than the block size $[r = (k + 1)]$. That is, the number of replications required is 6 for 25 treatments, 7 for 36 treatments, 8 for 49 treatments, and so on.

The main advantage of Balanced Lattices is that a large number of treatments may compare within relatively small blocks. Another advantage of Balanced Lattice Designs is that each pair of treatments is compared with the same degree of precision because each treatment occurs together in the same block with every other treatment an equal number of times (usually once). Hence, to obtain a Balanced Lattice, some restrictions on the number of treatments and the number of blocks in the design are required. Consequently, Balanced Lattices are not available for $60, 110, and\ 154$ treatments. The disadvantages of the

design are the limitation for the number of allowable treatments, block sizes and replication. The analysis also becomes more complex and the designs are more difficult to construct as the number of treatment increases.

Partially Balanced Lattice design on the other hand is a subclass of the incomplete block design which is similar to the Balanced Lattice design only that, it allows for a more flexible choice of the number of replications. Again, the Partially Balanced Lattice design requires that, the number of treatments must be a perfect square and that the block size must also be equal to the square root of these treatments number. In fact, any number of replications can be used in a Partially Balanced Lattice Design. They are well-known type of resolvable incomplete Block Designs. Partially Balanced Lattice Designs do not use all replication of the basic plan. For example, for 4X4 lattice designs, we need 5 replications for Balanced Design, but if we have only the first, first two, first three or first four replications these are Partially Balanced Designs.

With two replications, the partially Balanced Lattice Design is referred to as a simple lattice; with three replications, a triple lattice; with four replications, a quadruple lattice; and so on. In general, if the number of replication is m, it is called an m-ple Lattice. However, such flexibility in the choice of the number of replications results in a loss of symmetry in the arrangement of treatments over blocks (i.e., some treatment pairs never appear together in the same incomplete block). Consequently, the treatment pairs that are tested in the same Incomplete Block are compared with a level of precision that is higher than for those that are not tested in the same Incomplete Block.

Materials and Method

Construction of Replications of a partial Lattice Design

Alabi (2018) give alternative algorithm of constructing replications. This can be described as follows:

1. Write the treatment numbers $1, 2, \ldots, t$ consecutively in a square array of K rows and K columns to yield replicate 1 of the resolvable design, with rows constituting the blocks.
2. Transpose the rows and columns of replicate 1 to obtain replicate 2
3. Take the main right diagonal of replicate 2 to form the first row of replicate 3 and write the remaining elements in each column of replicate 2 in a cyclic order in the same column for replicate 3.

4. Repeat step 3 for replicate 3 to generate replicate 4.
5. Continue this process on the just generated replicate until the require replicates are obtained.

Example: Construct an optimal design for lattice square design with

$$t = 16 = 4^2, k = 4 \; and \; r = 4.$$

$$t = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

The result of the remaining three replications is shown in table below.

Table 1: Basic plan of a 4 x 4 Partially Balanced Lattice square Design involving sixteen Treatments in blocks of four Units and four Replications

Block No.	***Replication Number***			
	Rep. I	***Rep. II***	***Rep. III***	***Rep. IV***
1	*1 2 3 4*	*1 5 9 13*	*1 6 11 16*	*1 7 9 15*
2	*5 6 7 8*	*2 6 10 14*	*2 7 12 13*	*2 8 10 16*
3	*9 10 11 12*	*3 7 11 15*	*3 8 9 14*	*3 5 11 13*
4	*13 14 15 16*	*4 8 12 16*	*4 5 10 15*	*4 6 12 14*

Optimality Criteria

Bagmar *et al*, (2017) defined A-, D- and G-optimality as

A– optimality =

$$min_{xi=1,\ldots,n} trace(X'X)^{-1}.sum\big(diag((X'X)^{-1})\big) \qquad (16)$$

D–optimality is defined as $|X'X| = \frac{1}{|(X'X)^{-1}|} = \det\,(X'X)$ (17)

$$G - Optimality = \left(\frac{var(\hat{y}_x)}{N}\right) \qquad (18)$$

Efficiency Criteria

Kuhfeld, (2010) defined A-, D- and G-efficiency as

$$\text{A-efficiency} = 100 \times \frac{1}{Ntrace((X'X)^{-1})/p} = 100 \times \frac{P}{N \times A-optimality} \qquad (19)$$

$$D - Efficiency = 100\frac{(D - optimality)^{1/P}}{N} \quad (20)$$

G-efficiency= $$G - Efficiency = 100\sqrt{\frac{P/N_D}{\delta^2{}_M}} \quad (21)$$

Results and Discussion

Construction of Three Associate Class of Partially Balanced Lattice Design

Table 3: Layout of the Design

Block Number	*Treatments*			
1	*1*	*2*	*3*	*4*
2	*5*	*6*	*7*	*8*
3	*9*	*10*	*11*	*12*
4	*13*	*14*	*15*	*16*
5	*1*	*5*	*9*	*13*
6	*2*	*6*	*10*	*14*
7	*3*	*7*	*11*	*15*
8	*4*	*8*	*12*	*16*
9	*1*	*6*	*11*	*16*
10	*2*	*7*	*12*	*13*
11	*3*	*8*	*9*	*14*
12	*4*	*5*	*10*	*15*
13	*1*	*7*	*9*	*15*
14	*2*	*8*	*10*	*16*
15	*3*	*5*	*11*	*13*
16	*4*	*6*	*12*	*14*

Table 4: 1^{st}, 2^{nd} and 3^{rd} associates of all the treatments

Treatment No.	1^{st} *Associates*	2^{nd} *Associates*	3^{rd} *Associates*
1	*9*	*2, 3, 4, 5, 6, 7, 11, 13, 15, 16*	*8, 10, 12, 14*
2	*10*	*1, 3, 4, 6, 7, 8, 12, 13, 14, 16*	*5, 9, 11, 15*
3	*11*	*1, 2, 4, 5, 7, 8, 9, 13, 14, 15*	*6, 8, 10, 11*
4	*12*	*1, 2, 3, 5, 6, 8, 10, 14, 15, 16*	*7, 9, 11, 13*
5	*13*	*1, 3, 4, 6, 7, 8, 9, 10, 11, 15*	*2, 12, 14, 16*
6	*14*	*1, 2, 4, 5, 7, 8, 10, 11, 12, 16*	*3, 9, 13, 15*
7	*15*	*1, 2, 3, 5, 6, 8, 9, 11, 12, 13*	*4, 10, 14, 16*

8	16	2, 3, 4, 5, 6, 7, 9, 10, 14, 16	1, 11, 13, 15
9	1	3, 5, 7, 8, 10, 11, 12, 13, 14, 15	2, 6, 15, 16
10	2	4, 5, 6, 8, 9, 11, 12, 14, 15, 16	1, 3, 7, 13
11	3	1, 5, 6, 7, 9, 10, 12, 13, 15, 16	2, 4, 8, 14
12	4	2, 6, 7, 8, 9, 10, 11, 13, 14, 16	1, 3, 5, 15
13	5	1, 2, 3, 7, 9, 11, 12, 14, 15, 16	4, 6, 8, 10
14	6	2, 3, 4, 8, 9, 10, 12, 13, 15, 16	1, 5, 7, 11
15	7	1, 3, 4, 5, 9, 10, 11, 13, 14, 16	2, 6, 8, 12
16	8	1, 2, 4, 6, 10, 11, 12, 13, 14, 15	3, 5, 7, 8

RESULTS

From table (4) of the design 3 associate classes constructed with the following properties

Number of treatment in first column $(n_0) = 16$

Number of treatment in first associate $(n_1) = 1$

Number of treatment in the second associate $(n_2) = 10$

Number of treatment in the third associate $(n_3) = 4$

First associate $(\lambda_1) = 2$

Second associate $(\lambda_2) = 1$

Third associate $(\lambda_3) = 0$

These associate classes assist the researcher in the analysis of data from Lattice Design experiment.

Treatment 1

Treatment pairs $(1, 9)$**,** occurs twice in the blocks with treatment "1". These treatments are first associates of treatment "1" and $n_1 = 1$.

Treatment pairs $(1,2)$ $(1, 3)$ $(1,4)$ $(1,5)$ $(1,6)$ $(1,7)$ $(1,11)$ $(1,13)$ $(1,15)$ $(1,16)$. Occur once in the blocks with treatment "1". These treatments are second associates of treatment "1" and $n_2 = 10$.

Lastly, Treatment pairs $(1, 8)$ $(1, 10)$ $(1, 12)$ $(1, 14)$ which never occur in any block is called third associates of treatment "1" and $n_3 = 4$.

Treatment 2

Treatment pairs $(2, 10)$ occur twice in the blocks with treatment "2". These treatments are first associates of treatment "2" and $n_1 = 1$.

$Treatment\ pairs$ $(2,1)$ $(2,3)$ $(2,4)$ $(2,6)$ $(2,7)$ $(2,8)$ $(2,10)$ $(2,13)$ $(2,14)$ $(2,16)$ Occur once in the blocks with treatment "2". These treatments are second associates of treatment "1" and $n_2 = 10$.

Lastly, Treatment pairs $(2,5)$ $(2,9)$ $(2,11)$ $(2,15)$ which never occur in any block is called third associates of treatment "2" and $n_3 = 4$.

Treatment 3

Treatment pairs $(3,11)$ occurs twice in the blocks with treatment "3". These treatments are first associates of treatment "3" and $n_1 = 1$.

Treatment pairs $(3,1)$ $(3,2)$ $(3,4)$ $(3,5)$ $(3,7)$ $(3,8)$ $(3,9)$ $(3,13)$ $(3,14)$ $(3,15)$ occur once in the blocks with treatment"3". These treatments are second associates of treatment "3" and $n_2 = 10$.

Lastly, treatment pairs $(3,5)$ $(3,6)$ $(3,10)$ $(3,16)$ which never occur in any block is called third associates of treatment "2" and $n_3 = 4$.

These iterations are repeated until associates of all other treatments $(4,5,6,\dots,16)$ are obtained.

Design with one replication

$$\boldsymbol{X} = \begin{matrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{matrix}$$

Figure 1. The incidence matrix with 4 blocks, 16 treatments

Design with two replications

$$X = \begin{matrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{matrix}$$

Figure 2. The incidence matrix with 8 blocks, 32 treatments

Design with three replications

$$X = \begin{matrix}
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{matrix}$$

Figure 3. The incidence matrix with 12 blocks, 48 treatments

Design with four replications

$$X = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}$$

Figure 4. The incidence matrix with 16 blocks, 64 treatments.

$$X'X = \begin{matrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{matrix}$$

$$(x'x)^{-1} = \begin{matrix} 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 \end{matrix}$$

$$A - Optimality = min_{xi=1,\dots,n} trace = sum(diag(x'x)^{-1}) = 4 \quad (22)$$

$$A - Efficiency = 100 \times \frac{P}{N \times A - optimality} = 6.2500 \quad (23)$$

$$D - Optimality = min_{xi,i=1,\dots,n}\, max_{x \in X} |x'x| = 4.2950e + 009 \quad (24)$$

$$D = Efficiency 100 \frac{(D-optimality)^{1/P}}{N} = 100 \frac{(4.2950e+009)^{1/16}}{64} = 3.0038 \quad (25)$$

$$G - Optimality = min_{xi,i=1,\dots,n}\, max_{x \in X} \left(\frac{trace(x'x)}{P} \right) = \left(\frac{64}{16} \right) = 4 \quad (26)$$

$$G - Efficiency = 100 \sqrt{\frac{P/N_D}{\delta^2{}_M}} = 100 \sqrt{\frac{16/64}{64}} = 6.25 \quad (27)$$

Note: P is the number of blocks and N is the number of treatments.

Table 5: comparison of Optimality criteria

A – optimality	*D – optimality*	*G – optimality*
4	$4.2950e + 009$	4

Table 6: comparison of Efficiency criteria

A – Efficiency	*D – Efficiency*	*G – Efficiency*
6.25	3.0038	6.25

Conclusions

The research has carefully constructed 3 associate classes of a Partially Balanced Lattices Design of 16 treatments with four replications. In this manner, if treatments pairs occur twice in the blocks are said to be first associate (λ_1) and if its occur once in the blocks are said to be second associates (λ_2) but if the treatment pairs did not occur at all in the blocks are said to be third associate (λ_3).

The research has demonstrated how good a design is with respect to information matrix in four replications of Partial Lattices. Based on the analysis D-criteria show better result followed by A-criteria and G-criteria respectively.

Considering the efficiency of the design, the results of A-, D- and G- efficiency are 6.25%, 3.0038%, *and* 6.25% respectively. D-efficiency is better when compared to A- and G- efficiency because it was noted that the Efficiencies that are not near 100 may perfectly satisfactory and the more the efficient design is the more it tends toward balance and orthogonality.

In experimental design, a good criterion for one design may not be the best in another design. It has shown that the Partial Lattice possesses D- optimality criteria which advices experimenter to employ any of the designs. Therefore, the Optimality and the Efficiency of the Partially Balanced Lattice Design are independent of the number of the replications.

Recommendation

It recommends here that for studies in Lattice Designs,

i. D- Optimality is better when a researcher is interested in Square Lattice Designs.

ii. Information based criteria was used. However, distance- based criteria, compound design criteria and other criteria are recommended.

iii. Extension to four associate class designs recommended.

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