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DISCRIMINATION BETWEEN 2K AND 3K FACTORIAL DESIGNS USING OPTIMALITY BASED CRITERION

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ABSTRACT

This research aimed at identifying the point k in 2^K and 3^K designs where $k = 2, 3$for which optimality obtained using design optimality criteria. The codes ± 1 used for 2^Kfactorial designs while $\pm 1,\ 0$ and $0, 1, 2$ provided the elements of the design matrix for 3^K factorial designs was used to obtain the design matrix. While, A- and D- optimality criteria for each of the Designs model were utilized in the analysis with the aid of MATLAB version 7.5.0.345 (R2007) to determine the optimal values and the efficiency of each model. From our results, it observed that the 3^Kdesigns produced efficient estimates of the model parameters when A- and D- Optimality criteria are considered. Therefore, the 3^K factorial designs are more efficient than 2^Kdesigns for all values of k under consideration using design efficiency criteria. It recommended that for studies in factorial designs, 3^K design is suitable in both A- and D- optimality criteria.

Keywords: *optimality criteria, efficiency criteria,*

INTRODUCTION

Many experiments in industrial research and development involve studying the effect of several factors on one or more responses. Factorial experiment is an experiment whose design consists of two or more factors, each with levels, and whose experimental units take on all possible combinations of these levels across all such factors. A factorial design is also call crossed design. Such an experiment allows the investigator to study the effect of each factor level and interactions on the response variable.

For the vast majority of factorial experiments, each factor has only two levels. For example, with two factors each taking two levels, a factorial experiment would have four treatment combinations in total and is usually referred to as 2^2 factorial designs. likewise, with two factors each taking three levels, a factorial experiment would have nine treatment combinations in total and is usually referred to as 3^2 factorial designs. If the number of combinations in a factorial design is too high to be logically feasible, a fractional factorial design may be employed in which some of the possible combinations(usually at least half) are omitted.

Some of the most important contributions to the theory and practice of statistical inference in the twentieth century have been those in experimental design. Most of the early development was stimulated by applications in agriculture.

The use of experiment design methods in the chemical industry was promoted in the 1950s by the extensive works of box and his collaborators on response surface designs. Over the past years, there has been a tremendous increase in the application of experimental design techniques in industry. This is due largely to the increased emphasis on quality improvement and the important role played by statistical methods. In general, design of experiments in particular. In Japanese industry, the work of Japanese quality consultant G. Taguchi on robust design for variation reduction has shown the power of experimental design techniques for quality improvement.

Experimental design techniques are also becoming popular in the area of Computer-aided design and engineering using Computer/Simulation Models, including applications in manufacturing(automobile and Semi-Conductor Industries), as well as in the nuclear Industry(Conover and Iman, 1980). Statistical issues in the design and analysis of Computer/Simulation experiments are discussed in sacks et al.(1989).

Factorial design plays a fundamental role in efficient and economic experimentation with multiple input variables and is extremely popular in various fields of application, including engineering, agriculture, medicine and sciences. Scientist wishing to understand the effect of two or more independent variables upon a single dependent variable often uses a factorial design. The goal of many scientific and industrial experiments is to study a process that depends on several qualitative factors. This involves studying the effect of several factors on one or more responses.

Factorial designs are extremely useful to psychologist and field scientist as a preliminary study, allowing to judge whether there is a link between variables whilst reducing the possibility of experimental error and confounding variables. The factorial design as well as simplifying the process and making research cheaper, allows many levels of analysis as well as highlighting the relationships between variables, it also allows the effects of manipulating a single variable to be isolated and analyze singly.

Factorial designs are form of true experiment, where multiple factors (The researchers-Controlled independent variables)manipulated or allowed to vary and they provide researchers two main advantages. First, they allow researchers to examine the main effect of two or more individual independent variables simultaneously. Second, they allow researchers to detect interactions among variables. An interaction is when the effect of one variable varies according to the levels of another variable. Such interactions can only be detected when the variables are examined in combination. When using a factorial design, the independent variable referred to as a factor and the different values of a factor referred to as levels.

The main disadvantage of factorial design is the difficulty of experimenting with more than two factors, or many levels. A factorial design has to be plan meticulously, as an error in one of the levels or in the general operations, will jeopardize a great amount of work.

Other than these slights detractions, a factorial design is a mainstay of many scientific disciplines, delivering great results in the field.

Methodology

The 2^k Model

In factorial design we use large letters to denote the effects, i.e.*A*=A effect. AB= interaction of A and B.

For $k = 2$ that is 2^2factorial designs, the notation is as follows:

$$a0b0, a1b0\ a0b1, a1b1$$

Where

$a0b0$ =(1) when each of the factors occurs at lower level,

$a1b0 =$ i.e. a is at high level while b is at low level,

$a0b1 =$ i.e. a is at low level while$b$ is at high level,

$a1b1 =$ i. eBoth ab factors is at high level.

Treatment combinations are $(1), a, b, ab$

Table 3.1.1: the algebraic sign for calculating the effect of $\mathbf{2^2}$ designs

$Treatment$		$FactorialEffect$		
$Combination$	I	A	B	AB
$(\mathbf{1})$	$+$	$-$	$-$	$+$
$\boldsymbol{a}$	$+$	$+$	$-$	$-$
$\boldsymbol{b}$	$+$	$-$	$+$	$-$
$\boldsymbol{ab}$	$+$	$+$	$+$	$+$

Representing it in matrix form + represents 1 and – represents -1(i.e. the high and the low level of each factor respectively).
Thus the design matrix for 2^2 is

$$X = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The linear statistical model for a 2^2 design is as follows:

$$y_{ijk} = \mu + \tau_i + \tau_j + \lambda_{ij} + \varepsilon_{ijk} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

Where

y_{ijk} Represent the observation taken under the i^{th} level of factor A and the j^{th} level of factor b in the K^{th} replicate

For $k = 3$ that is 2^3factorial designs, the notation is as follows:

$a_0b_0c_0, a_1b_0c_{0,}a_0b_1c_0, a_1b_1c_0, a_0b_0c_1, a_0b_1c_1, a_1b_1c_1$

Where

$000 = a_0b_0c_0$≡(1) when each of the factors occurs at lower level,
100 =100 =$a_1b_0c_0$=a i.e. a is at high level where b and c is at low level,
$010 = a_0b_1c_0$=b i.e. a and c is low level while b is at high level,
$110 = a_1b_1c_0 = ab$I.e. a and b is at high level,
$001 =$ c is at low level, $a_0b_0c_1 = c$ i.e. c is at high level
101 =a and b is at low level, $a_1b_0c_1 = ac$ i.e. a and c is at high level
$011 =$ b is at low level, $a_0b_1c_1 = bc$ i.e. a is at low level
111 =a, b and c is at high level, $a_1b_1c_1 = abc$i.e. both factors are at high level.

The Treatment combinations are: $(1), a, b, ab, c, ac, bc, abc$.

Table 3.1.2: The algebraic sign for calculating the effect of 2^3 designs

Treatment	**Factorial Effect**							
Combination	I	A	B	AB	C	AC	BC	ABC
1	+	−	−	+	−	+	+	−
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
c	+	−	−	+	+	−	−	+
ac	+	+	−	−	+	+	−	−
bc	+	−	+	−	+	−	+	−
abc	+	+	+	+	+	+	+	+

Thus the design matrix for 2^3 designs is

$$X = \begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The linear statistical model for a 2^3 design is as follows:

$$y_{ijkl} = \mu + \tau_i + \tau_j + \tau_k + (\lambda)ij + (\lambda)_{ik} + (\lambda)_{jk} + (\lambda)_{ijk} + \varepsilon_{ijkl} \begin{cases} i = 1, 2, \dots, a \\ j = 1, 2, \dots, b \\ k = 1, 2, \dots, c \\ l = 1, 2, \dots, n \end{cases}$$

Where

y_{ijkl} Represent the observation taken under the i^{th} level of factor A and the k^{th} level of factor B in the k^{th} level of factor C in the k^{th} replicate.

μ Represent the overall mean effect,

τ_i Represent the true effect of the i^{th} level of factor A,

τ_j Represent the true effect of the j^{th} level of factor B,

τ_r Represent the true effect of the k^{th} level of factor C
$(\lambda)_{ij}$ Represent the effect of the interaction between τ_i and τ_j
$(\lambda)_{ir}$ Represent the effect of interaction between τ_i and τ_r
$(\lambda)_{jr}$ Represent the effect of interaction between τ_j and τ_r
$(\lambda)_{ijr}$ Represent the effect of interaction between $\tau_i, \tau_j\ and\ \tau_r$
ε_{ijr} Represent the random error component.

3.2 The 3^k model

For k = 2 i.e.$3^2$the statistical model in regression form is given as follows

$$y(x_1x_2) = \beta_{00} + \beta_{10}x_1 + \beta_{01}x_2 + \beta_{11}x_1x_2 + \beta_{20}x_1^2 + \beta_{02}x_2^2 + \beta_{21}x_1^2x_2 + \beta_{22}x_1^2x_2^2 + \varepsilon$$

Where

$00 = \beta_{00}$ Represent both factors at low level denoted by (1),

10 = $\beta_{10}x_1$ Represent factor A at intermediate level denoted by A_L

$01 = \beta_{01}x_2$ Represent factor B at intermediate level denoted B_L

$11 = \beta_{11}x_1x_2$ Represent the interaction between factor A and B at intermediate level denoted by A_LB_L,

$20 = \beta_{20}x_1^2$ Represent factor A at high level denoted by A_Q,

$02 = \beta_{02}x_2^2$ Represent factor B at high level denoted by B_Q

$21 = \beta_{21}x_1^2x_2$ Represent the interaction between factor A at high level and factor B at intermediate level denoted byA_QB_L,

$12 = \beta_{12}x_1x_2^2$ Represent the interaction between factor A at intermediate level and factor B at high level denoted byA_LB_Q,

$22 = \beta_{22}x_1^2x_2^2$ Represent both factors at high level denoted byA_QB_Q.

$\varepsilon =$ Is the error component.

The standard order for a 3^2 design is$(1), A_L, A_Q, B_L, A_LB_L, A_QB_L, B_Q, A_LB_Q$

Table 3.2.1: Design matrix for $\mathbf{3^2}$ factorial experiments is as follows

$(X1, X2)$	β_{00}	β_{01}	β_{02}	β_{10}	β_{11}	β_{12}	β_{20}	β_{21}	β_{22}
$(-1,-1)$	1	-1	1	-1	1	-1	1	-1	1
$(-1,0)$	1	0	0	-1	0	0	1	0	0
$(-1,1)$	1	1	1	-1	-1	-1	1	1	1

$(\mathbf{0}, -\mathbf{1})$	1	−1	1	0	0	0	0	0	0
$(\mathbf{0}, \ \mathbf{0})$	1	0	0	0	0	0	0	0	0
$(\mathbf{0}, \mathbf{1})$	1	1	1	0	0	0	0	0	0
$(\mathbf{1}, -\mathbf{1})$	1	−1	1	1	−1	1	1	−1	1
$(\mathbf{1}, \mathbf{0})$	1	0	0	1	0	0	1	0	0
$(\mathbf{1}, \mathbf{1})$	1	1	1	1	1	1	1	1	1

Thus, the design matrix for $3^2$is

$$X = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

For k=2 i.e. 3^2 the statistical model in regression form is given as follows:Then the least squares estimates of the parameters μ, $(T1)_1$, $(T1)_2$, $(T1)_3$, $(T2)_1$, $(T2)_2$, $(T1)_3$,and $(T2)_4$are

$\hat{\mu} = \bar{Y}_{..}$ 1

$(\hat{T}1)_1 = \bar{Y}_{1.} - \bar{Y}_{..}$

$(\hat{T}1)_2 = \bar{Y}_{2.} -_3 = \bar{Y}_{3.} - \bar{Y}_{..}$

$(\hat{T}2)_1 = \bar{Y}_{.1} - \bar{Y}_{..}$

$(\hat{T}2)_2 = \bar{Y}_{.2} - \bar{Y}_{..}$

$(\hat{T}2)_3 = \bar{Y}_{.3} - \bar{Y}_{..}$

$(\hat{T}2)_4 = \bar{Y}_{.4} - \bar{Y}_{..}$

The standard order for a 3^3 design is

$(1), A_L, A_Q, B_L, A_LB_L, A_QB_L, B_Q, A_LB_Q, A_QB_Q, C_L, A_LC_L, A_QC_L, B_LC_L, A_LB_LC_L,$

$A_QB_LC_L, B_QC_{L,}A_LB_QC_L, A_QB_QC_L, C_{Q,}A_LC_Q, A_QC_Q, B_LC_Q, A_LB_LC_Q, A_QB_LC_Q$

$, B_QC_Q, A_LB_QC_Q, A_QB_QC_Q$

Coded as
$000, 100, 200, 010, 110, 210, 020, 120, 220, 001, 101, 201, 011,111, 211, 021, 121, 221$
$,002, 102,202, 012, 112, 212, 022, 122 \ and \ 222 \ respectively$

Table 3.2.2: Design matrix for 3^3 factorial experiments.

Run	A	B	C	AB	AB^2	AC	AC^2	BC	BC^2	ABC	ABC^2	AB^2C	AB^2C^2
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	0	0	1	2	1	2	1	2	1	2
3	0	0	2	0	0	2	1	2	1	2	1	2	1
4	0	1	0	1	2	0	0	1	1	1	1	2	2
5	0	1	1	1	2	1	2	2	0	2	0	0	1
6	0	1	2	1	2	2	1	0	2	0	2	1	0
7	0	2	0	2	1	0	0	2	2	2	2	1	1
8	0	2	1	2	1	1	2	0	1	0	1	2	0
9	0	2	2	2	1	2	1	1	0	1	0	0	2
10	1	0	0	1	1	1	1	0	0	1	1	1	1
11	1	0	1	1	1	0	0	1	2	2	0	2	0
12	1	0	2	1	1	2	2	2	1	0	2	0	2
13	1	1	0	2	0	1	1	1	1	2	2	0	0
14	1	1	1	2	0	0	0	2	0	0	1	1	2
15	1	1	2	2	0	2	2	0	2	1	0	2	1
16	1	2	0	0	2	1	1	2	2	0	0	2	2
17	1	2	1	0	2	0	0	0	1	1	2	0	1
18	1	2	2	0	2	2	2	1	0	2	1	1	0
19	2	0	0	2	2	2	2	0	0	2	2	2	2
20	2	0	1	2	2	1	1	1	2	0	1	0	1
21	2	0	2	2	2	0	0	2	1	1	0	1	0
22	2	1	0	0	1	2	2	1	1	0	0	1	1
23	2	1	1	0	1	1	1	2	0	1	2	2	0

24	2	1	2	0	1	0	0	0	2	2	1	0	2
25	2	2	0	1	0	2	2	2	2	1	1	0	0
26	2	2	1	1	0	1	1	0	1	2	0	1	2
27	2	2	2	1	0	0	0	1	0	0	2	2	1`

Thus the design matrix for 3^3 factorial is

$$X = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&1&2&1&2&1&2&1&2\\ 0&0&2&0&0&2&1&2&1&2&1&2&1\\ 0&1&0&1&2&0&0&1&1&1&1&2&2\\ 0&1&1&1&2&1&2&2&0&2&0&0&1\\ 0&1&2&1&2&2&1&0&2&0&2&1&0\\ 0&2&0&2&1&0&0&2&2&2&2&1&1\\ 0&2&1&2&1&1&2&0&1&0&1&2&0\\ 0&2&2&2&1&2&1&1&0&1&0&0&2\\ 1&0&0&1&1&1&1&0&0&1&1&1&1\\ 1&0&1&1&1&2&0&1&2&2&0&2&0\\ 1&0&2&1&1&0&2&2&1&0&2&0&2\\ 1&1&0&2&0&1&1&1&1&2&2&0&0\\ 1&1&1&2&0&2&0&2&0&0&1&1&2\\ 1&1&2&2&0&0&2&0&2&1&0&2&1\\ 1&2&0&0&2&1&1&2&2&0&0&2&2\\ 1&2&1&0&2&2&0&0&1&1&2&0&1\\ 1&2&2&0&2&0&2&1&0&2&1&1&0\\ 2&0&0&2&2&2&2&0&0&2&2&2&2\\ 2&0&1&2&2&0&1&1&2&0&1&0&1\\ 2&0&2&2&2&1&0&2&1&1&0&1&0\\ 2&1&0&0&1&2&2&1&1&0&0&1&1\\ 2&1&1&0&1&0&1&2&0&1&2&2&0\\ 2&1&2&0&1&1&0&0&2&2&1&0&2\\ 2&2&0&1&0&2&2&2&2&1&1&0&0\\ 2&2&1&1&0&0&1&0&1&2&0&1&2\\ 2&2&2&1&0&1&0&1&0&0&2&2&1 \end{pmatrix}$$

Assuming the interaction between three factors has the form

$$(\mathrm{T12})_{ij} = \lambda_{12}(\mathrm{T1})_i\,(\mathrm{T2})_{ij},$$

$$(\mathrm{T13})_{ij} = \lambda_{13}(\mathrm{T1})_i\,(\mathrm{T3})_{ir},$$

$$(\text{T23})_{ij} = \lambda_{23}(\text{T2})_j\,(\text{T3})_r, and$$

$$(\text{T123})_{ij} = \lambda_{123}(\text{T1})_i\,(\text{T2})_j\,(\text{T3})_r,$$

Then the least squares of the parameters
$(T1)_1$, $(T1)_2$, $(T1)_3$, $(T2)_1$, $(T2)_2$, $(T3)_1$,and $(T3)_2$ are

$$\hat{\mu} = \bar{Y}$$

RESULT

This section contains the result of data analysis of 2^K and 3^Kfactorial designs without replicationfor $K = 2{,}3$.Using MATLAB.
The result of 2^2 factorial designs is as follows

$$X = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$X'X = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

A-optimality= Trace of the matrix = $sum(diag\ (X'X) = 16$
D-optimality= $\det\,(X'X) = 256$
P=4; N=4;

Where
N= number of columns and P =number of parameter in the model

A-efficiency= $100(P/(N * A - \ Optimality) = \ 6.2500$

D-efficiency= $100\left(\frac{D-Opimality^{\frac{1}{P}}}{N}\right) = 100$

The result of 2^3 factorial designs is as follows

$$X = \begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$X'X = \begin{pmatrix} 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{pmatrix}$$

A-optimality=Trace of the matrix = $sum(diag(X'X) = 64$

D-optimality= $\det(X'X) = 16777216$

P=8; N=8;Where

p = number of parameter in the model and N = number of columns

A-efficiency= $100\left(\frac{P}{N*A-optimality}\right) = 1.5625$

D-efficiency== $100\left(\frac{D-Opimality^{\frac{1}{P}}}{N}\right) = 100$

Table 4.1.1: optimality Table of$\mathbf{2^k}$ factorial designs

	A-Optimality	D-Optimality
$\mathbf{2^2}$	16	256
$\mathbf{2^3}$	64	16777216

Table 4.1.2: Efficiency Table of$\mathbf{2^k}$ factorial designs

	A-Efficiency	D-Efficiency
$\mathbf{2^2}$	6.2500	100
$\mathbf{2^3}$	1.5625	100

The result of 3^2 factorial designs is as follows

$$X = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$X'X = \begin{pmatrix} 9 & 0 & 6 & 0 & 0 & 0 & 6 & 0 & 4 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 6 & 0 & 6 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 \\ 6 & 0 & 4 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 4 & 0 & 4 & 0 & 0 & 0 & 4 & 0 & 4 \end{pmatrix}$$

A-optimality= $Trace = sum\left(diag\ (X'X)^{-1})\right) = 49$
D-optimality= $\det\ (X'X)\ = 1024$

P= 9; N= 9; Where

P= number of parameter in the model and N = number of columns

A-efficiency= $100(P/(N * A - Optimality) = 2.0408$

D-efficiency= $100\left(\frac{4096)^{\frac{1}{9}}}{9}\right) = 24.0013$

The result of 3^3 factorial designs is as follows

$$X = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
0 & 0 & 2 & 0 & 0 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\
0 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 \\
0 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 0 & 2 & 0 & 0 & 1 \\
0 & 1 & 2 & 1 & 2 & 2 & 1 & 0 & 2 & 0 & 2 & 1 & 0 \\
0 & 2 & 0 & 2 & 1 & 0 & 0 & 2 & 2 & 2 & 2 & 1 & 1 \\
0 & 2 & 1 & 2 & 1 & 1 & 2 & 0 & 1 & 0 & 1 & 2 & 0 \\
0 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & 2 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 2 & 0 & 1 & 2 & 2 & 0 & 2 & 0 \\
1 & 0 & 2 & 1 & 1 & 0 & 2 & 2 & 1 & 0 & 2 & 0 & 2 \\
1 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 0 & 0 \\
1 & 1 & 1 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 1 & 1 & 2 \\
1 & 1 & 2 & 2 & 0 & 0 & 2 & 0 & 2 & 1 & 0 & 2 & 1 \\
1 & 2 & 0 & 0 & 2 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 2 \\
1 & 2 & 1 & 0 & 2 & 2 & 0 & 0 & 1 & 1 & 2 & 0 & 1 \\
1 & 2 & 2 & 0 & 2 & 0 & 2 & 1 & 0 & 2 & 1 & 1 & 0 \\
2 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 2 \\
2 & 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & 0 & 1 \\
2 & 0 & 2 & 2 & 2 & 1 & 0 & 2 & 1 & 1 & 0 & 1 & 0 \\
2 & 1 & 0 & 0 & 1 & 2 & 2 & 1 & 1 & 0 & 0 & 1 & 1 \\
2 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 2 & 0 \\
2 & 1 & 2 & 0 & 1 & 1 & 0 & 0 & 2 & 2 & 1 & 0 & 2 \\
2 & 2 & 0 & 1 & 0 & 2 & 2 & 2 & 2 & 1 & 1 & 0 & 0 \\
2 & 2 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 \\
2 & 2 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 2 & 1
\end{pmatrix}$$

$$X'X = \begin{pmatrix} 45 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 \\ 27 & 45 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 \\ 27 & 27 & 45 & 27 & 27 & 30 & 27 & 27 & 27 & 27 & 27 & 27 & 27 \\ 27 & 27 & 27 & 45 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 \\ 27 & 27 & 27 & 27 & 45 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 \\ 27 & 27 & 30 & 27 & 27 & 45 & 42 & 27 & 27 & 27 & 27 & 27 & 27 \\ 27 & 27 & 27 & 27 & 27 & 42 & 45 & 27 & 27 & 27 & 27 & 27 & 27 \\ 27 & 27 & 27 & 27 & 27 & 42 & 45 & 27 & 27 & 27 & 27 & 27 & 27 \\ 27 & 27 & 27 & 27 & 27 & 27 & 27 & 45 & 27 & 27 & 27 & 27 & 27 \\ 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 45 & 27 & 27 & 27 & 27 \\ 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 45 & 27 & 27 & 27 \\ 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 45 & 27 & 27 \\ 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 27 & 45 \end{pmatrix}$$

A-optimality=$Trace = sum\big(diag(x'x)\big) = 585$

D-optimality= $det(x'x) = 1.0990 + 017$

P= 13;N= 27; Where

P =number of parameter(s) in the model and N = number of columns

A-efficiency= $100\ \left(\frac{p}{N*A\ optimality}\right)$ =A-efficiency $= 0.0823$

D-efficiency = $100\left(\frac{D\ 0ptimality)^{\frac{1}{p}}}{N}\right) = 75.7673$

Table 4.2.1: optimality Table of $\mathbf{3^k}$ factorial designs

	A-Optimality	D-Optimality
$\mathbf{3^2}$	49	1024
$\mathbf{3^3}$	585	$1.0990e + 017$

Table 4.2.2: Efficiency Table of$\mathbf{3^k}$ factorial designs

	A-Efficiency	D-Efficiency
$\mathbf{3^2}$	2.0408	24.0013
$\mathbf{3^3}$	0.0823	75.7673

Table 4.3: Comparison of optimality and Efficiency of 2^k and 3^k factorial designs

Model	A-Optimality	D-Optimality	A-efficiency	D-efficiency
$\mathbf{2^2}$	16	256	6.25	100
$\mathbf{2^3}$	64	16777216	1.5625	100
$\mathbf{3^2}$	49	1024	2.0408	24.0013
$\mathbf{3^3}$	585	$1.0990e + 017$	0.0823	75.7673

Relative efficiency

Relative efficiency of$2^k and\ 3^k$ factorial designs with respect to A-efficiency

Relative efficiency of $2^2 and\ 3^2$

RE ($\hat{y}_2$: $\hat{y}_1$) = $2.0408/6.25\,00 = 0.3265$

Since RE $(\hat{y}2{:}\,\hat{y}1) < 1$ it means that $\hat{y}1$is more efficient than $\hat{y}2$

Relative efficiency of $2^3 and\ 3^3$

RE$(\hat{y}2{:}\,\hat{y}1) =\ 0.0823/1.5625\ = 0.0526$

Since RE $(\hat{y}2{:}\,\hat{y}1) < 1$ it means that $\hat{y}1$is more efficient than $\hat{y}2$

Where$\hat{y}1 = 2^2$ and $\hat{y}2 = 3^2$

Relative efficiency of $2^k and\ 3^k$ factorial designs with respect to D-efficiency

Relative efficiency of $2^2 and\ 3^2$

RE $(\hat{y}2:\hat{y}1) = 24.0013/100 = 0.2400$
Since RE $(\hat{y}2:\hat{y}1) < 1$ it means that $\hat{y}1$is more efficient than $\hat{y}2$
Relative efficiency of $2^3 and\ 3^3$
RE $(\hat{y}2:\hat{y}1) 75.7673/100 = 0.7577$
Since RE$(\hat{y}2:\hat{y}1) < 1$ it means that $\hat{y}1$is more efficient than $\hat{y}2$
Where $\hat{y}1 = 2^3$and$\hat{y}2 = 3^3$

CONCLUSION

Table 3.1.1 and Table 3.1.2 shows how the designs matrix of 2^k factorial designs were constructed. Likewise, the design matrices of 3^k factorial designswere presented in Table 3.2.1 and Table 3.2.2 respectively.
From Table 4.3.1, we notice that the trace and determinant of $(x'x)$ in 2^kdesign is less than that of 3^k.this implies that 2^k is optimal for all values of k considered.
Also from Table 4.3.1, we notice that $2^k design$ is more efficient than 3^kdesign since its values approaches 100%for all values of k considered for A- efficiency and D- efficiency respectively.
When comparing the relative A-efficiency of model one (2^k factorial designs) and model two (3^k factorial designs). RE$(\hat{y}2:\hat{y}1)$ from the analysis, we noticed that$\hat{y}1$ is more efficient than $\hat{y}2$since RE $(\hat{y}2:\hat{y}1) < 1$. In the same vein, we compared D-efficiency of model one and model two from the analysis, we notice that $\hat{y}1$ is more efficient than $\hat{y}2$ sinceRE $(\hat{y}2:\hat{y}1) < 1$.
Also from the analysis, it noticed that the lower the number of factors in a corresponding model, the higher the efficient it becomes. Therefore, 2^kfactorial designs are more efficient than 3^kfactorial designs since the number of factors of 3^kdesigns is much larger than that of 2^k.this is to say that2^k design produced efficient estimates of the model when A- and D- efficiency isconsidered.

Recommendation

It recommends here that for studies in factorial designs,
Both A- and D-Efficiency arebetter when a researcher is interested in 2^kFactorial designs.

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