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INTRINSIC GROWTH RATE AND CUMMULATIVE RATE OF EMISSION OF TOXICANTS OF AN ENVIRONMENTAL-ECOLOGICAL INTERACTION: A MODEL INVESTIGATION

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Abstract

In this study, stability analysis of a mathematical model of an environmental-ecological interaction, using a computational approach, was considered. A continuous dynamical system of nonlinear first-order ordinary differential equations was used in the simulation. The stability theory of nonlinear differential equations was applied to analyse and predict the behaviour of the system. It was assumed that the competing species and their resources are affected by pollutants emitted into the environment from different sources. Steady-state solutions were obtained and were characterized using linearization method. The stability analysis of steady-state solutions of the system was considered and then the impact of the intrinsic growth rate and the cumulative rate of emission of the same toxicant on the stability of the dynamical system was evaluated, using a computational approach. The result shows that irrespective of the variation of the intrinsic growth rate, a_1, the stability of the system would depend on the cumulative rate of emission of the same toxicant, Q, into the environment from different sources.

Keywords: *Intrinsic growth rate, interaction, cumulative rate of emission, toxicants, stability, model investigation.*

Introduction

Naturally, humans interact with their environment to get food, water, fuel, medicine and space for their survival. In some cases, the competition between two species leads to a steady-state of coexistence, while in other cases the competition results in the eventual extinction of one of the species [Brannan and Boyce (2010); Lannel and Pugliese (2014)].

Advancement in science and Technology has made it possible for humans to exploit the environment for their benefit, but toxicants have also been introduced causing environmental damage. Human activities, such as felling of trees and gas flaring have altered the environment. Clean water, a good climate, fertile soils are a part of the environment that enables humans to live and flourish. However, an infertile land, contaminated water, polluted air and hot climate make it difficult for human beings to survive. The effect of environmental pollution on humans is quite revealing, affecting all human activities.

As a result of technological advancement and industrialization, large quantity of toxicants is released into both the aquatic and terrestrial environments. The skill with which human species have exploited human resources to fulfill their needs is dazzling. The accelerated extent of human actions destabilizes long-standing ecological balance. The dangers of local refining of fossil fuels and local production of fish feed cannot be adequately quantified. For instance, the local oil refiner produces pollutants in the form of gases and particulate matters by burning fossil fuels while the fish farmer discharges contaminated water, which get transformed in the environment into harmful substances by chemical reaction. These pollutants seriously affect the survival of living organisms in the same environment as well as their resources. The stability of the system is of great concern to researchers when competing species and their resources are both affected by pollutants released into the environment from various sources.

Environment refers to our physical surroundings and the characteristics of the place we live; it involves all land, sea and atmosphere. Ecology is an aspect of the natural sciences that utilized mathematical methods in exploring organisms in their environment. It involves investigation of the interactions among organisms and their environment.

The knowledge of a dynamical system is one of the oldest hallmarks in the history of an aspect of mathematical theory called stability theory and differential equations. A computational approach involves a computing approach mainly through the analysis of mathematical models implemented on a computer. For a dynamical system of nonlinear first-order ordinary differential equations, it takes a longer calculation time to study the qualitative behaviour of the system using an analytical approach. However, the application of a computational approach, by a MATLAB numerical simulation scheme, to study the qualitative behaviour of the system, gives a shorter calculation time and more efficient results (Ford *et al.*, 2010). It is against this background that the technique of using a numerical scheme, such as MATLAB ODE45, has been considered.

Mishra *el al.* (2015) have used stability theory of differential equations to study the effects of toxicant on the stability and harvesting of marine species. They proved the existence of interior steady-state and also showed that increasing the emission rates of pollutants decreases the population of species.

Similarly, Hallam *et al.* (1983) have proposed and analyzed a differential equation model to study the effect of a toxicant present in the environment on a single species population. They discovered that the growth rate density of the species diminishes linearly with the concentration of toxicant, but the corresponding carrying capacity does not depend upon the concentration of toxicant present in the environment. Also, Samanta and Matti (2004) in their study of the effect of toxicant on a single biological species have considered instantaneous spill, constant emission and rapidly fluctuating random emission of toxicant into the environment. Their findings revealed that the toxicant concentration emitted instantaneously would not be sufficient to kill the population whereas the constant emission makes the population to settle down to steady state while the stability of the system would not depend on the washout rate of toxicant from the environment. Furthermore, Dubey and Hussain (2006) have proposed and analyzed a differential equation model for the survival of a resource-biomass-dependent single species population that is affected by a pollutant present in the environment. Their result shows that, an appropriate level of the resource biomass can be maintained and the survival of the species may be ensured.

Misra (2014) considered the effect of toxicants in a three species food chain system with food limited growth of prey population. It was revealed that, if the toxicant input rate is increased, then all the three species may tend to extinction while, if the toxicant input rate is decreased, the steady-states of all the three species increase.

Shukla *et al.* (2001) have investigated the survival of two competing species in a polluted environment by using the method of local stability analysis. They showed that the competitive outcomes may be affected in the presence of a toxicant. Also, Shukla *et al.* (2009) considered the survival of a resource-dependent population to find out the effect of toxicant emitted from external sources as well as formed by its precursors. Their findings revealed that the densities of resource and the population decrease as the cumulative emission rate of environmental toxicant increases.

Agarwal and Devi (2010) have proposed and studied a nonlinear differential equation model for the survival of biological species affected by toxicants present in the environment. Their analysis reveals that, as the emission rate of toxicants in the environment increases, the density of biological species decreases. They emphasized that the biological species may even become extinct if the rate of emission of pollutants increases continuously.

Furthermore, Agarwal and Devi (2011) systematically constructed a resource-dependent competition model by considering the combined effect of toxicants that were emitted from external sources as well as formed by precursors of competing species. In this important interdisciplinary study, the mathematical techniques of the equilibrium analysis and the local stability analysis with a few numerical simulations were utilized to analyze this continuous system of nonlinear first order ordinary differential equations. A key contribution of this study is that increase in the cumulative rate of emission of the toxicant from external sources as well as its rate of formation from precursors decrease the equilibrium density of both competing species and its resource.

In order to study the existence and survival of resource-dependent species living in a polluted environment, Dubey and Hussain (2003) studied nonlinear differential equation models for the survival of two resource-dependent competing species in an industrial environment. They showed that an appropriate level of resource biomass can be maintained to ensure the survival of species if suitable efforts to conserve the resource biomass and to control the undesired level of industrialization pressure are made.

Furthermore, Naresh *et al.* (2006) have proposed and analyzed a nonlinear differential equation model to study the effect of intermediate toxic product on the survival of a resource-dependent species in a polluted environment. The analysis of the model revealed that with increase in the cumulative rate of emission of toxicants in the atmosphere, the densities of resource biomass and the species dependent on it decreases and attain their lowest steady-state. It further shows that if the rate of emission is large enough, the resource biomass may become extinct under certain conditions and the species dependent on it may not survive. Similarly, Naresh *et al.* (2012) have used the local stability analysis method to study the effect of intermediate toxic product on the resource biomass and on the survival of species dependent on it. Their analysis shows that as the rate of emission of toxicant increases, the steady-state of biomass decreases.

In summary, other models that other researchers have used to analyze the stability of a dynamical system have been found, but the analysis, by numerical methods, of the stability properties of steady-state solution of an environmental-ecological system is not popular.

Materials and Method

The physical problem under consideration consists of two competing resource-dependent biological species, the local oil refiner and fish farmer, where both competing species as well as their resources are affected by toxicants emitted into the environment from different sources. To achieve the objectives of this work, the following five-dimensional system of nonlinear first-order ordinary differential equations, as given by Agarwal and Devi (2011), has been considered:

$$\frac{dx_1(t)}{dt} = a_1x_1 - a_2x_1^2 - \alpha x_1x_2 + \alpha_1x_1R - k_1\delta_1x_1T, \qquad x_1(0) = x_{10} \geq 0, \qquad (1)$$

$$\frac{dx_2(t)}{dt} = b_1x_2 - b_2x_2^2 - \beta x_1x_2 + \beta_1x_2R - k_2\delta_2x_2T, \qquad x_2(0) = x_{20} \geq 0, \qquad (2)$$

$$\frac{dR(t)}{dt} = c_1R - c_2R^2 - \alpha_1x_1R - \beta_1x_2R - k\gamma RT, \qquad R(0) = R_0 \geq 0, \qquad (3)$$

$$\frac{dP(t)}{dt} = \eta x_1 + \eta x_2 - (\lambda_0 + \theta)P, \qquad P(0) = P_0 \geq 0, \qquad (4)$$

$$\frac{dT(t)}{dt} = Q + \mu\theta P - \delta_0T - \delta_1x_1T - \delta_2x_2T - \gamma RT, \qquad T(0) = T_0 \geq 0, \qquad (5)$$

where x_1 and x_2 are the densities of the first and second competing species respectively, R is the density of resource biomass, P is the cumulative concentration of precursors produced by species forming the toxicant, T is the concentration of the same toxicant in the environment under consideration, Q is the cumulative rate of emission of the same toxicant into the environment from various external sources, a_1 and b_1 are the intrinsic growth rates of the first and second species, respectively, a_2 and b_2 denote the intra-competition coefficients of the first and second species respectively, $\alpha,\ \beta$ represent the inter-competition coefficients of first and second species respectively, α_1 and β_1 are the growth rate coefficients of first and second species, respectively due to resource biomass. k_1, k_2 and k are fractions of the assimilated amount of toxicants directly affecting the growth rates of densities of competing species and resource biomass, η is the growth rate coefficient of the cumulative concentration of precursors. λ_0 is the depletion rate coefficient of precursors due to natural factors whereas θ is the depletion rate coefficient caused by its transformation into the same toxicant of concentration T. μ is the rate of formation of the toxicant from precursors of competing species. δ_1, δ_2 and γ are the rates of depletion of toxicant in the environment due to uptake of toxicant by species and their resource biomass, respectively.

It is assumed that the resource biomass grows logistically with the supply rate of the external resource input to the system by a constant c_1 and its density reduces due to certain degradation factors present in the environment at a rate c_2. Furthermore, it is assumed that the toxicant in the environment is washed out or broken down with rate δ_0.

According to linear stability analysis, a steady-state solution is stable if all the eigenvalues of the Jacobian, evaluated at that steady-state solution, have negative real parts. However, if at least one of the eigenvalues has a positive real part, the system is unstable (Ford *et al.* 2010).

In calculating the eigenvalues, the following mathematical structure is considered:

$$\frac{dx_1(t)}{dt} = F_1(x_1, x_2, R, P, T), \qquad x_1(0) = x_{10} \geq 0, \qquad (6)$$

$$\frac{dx_2(t)}{dt} = F_2(x_1, x_2, R, P, T), \qquad x_2(0) = x_{20} \geq 0, \qquad (7)$$

$$\frac{dR(t)}{dt} = F_3(x_1, x_2, R, P, T), \qquad R(0) = R_0 \geq 0, \qquad (8)$$

$$\frac{dP(t)}{dt} = F_4(x_1, x_2, R, P, T), \qquad P(0) = P_0 \geq 0, \qquad (9)$$

$\frac{dT(t)}{dt} = F_5(x_1, x_2, R, P, T), \qquad T(0) = T_0 \geq 0,$ (10)

where it is assumed that F_1, F_2, F_3, F_4 and F_5 are continuous functions of the variables x_1, x_2, R, P and T.

Following Yan and Ekaka-a (2011), Leticia and Oleka (2016), George *et al.* (2018), George (2019) and several other mathematical literatures which cannot be cited at this stage, at a steady-state solution all rates of change are simultaneously equal to zero. Hence, equations (6) – (10) become

$\frac{dx_1}{dt} = \frac{dx_2}{dt} = \frac{dR}{dt} = \frac{dP}{dt} = \frac{dT}{dt} = 0.$ (11)

Therefore, the steady-state solution at an arbitrary point $(x_{1e}, x_{2e}, R_e, P_e, T_e)$ satisfies

$F_1(x_{1e}, x_{2e}, R_e, P_e, T_e) = 0,$ (12)

$F_2(x_{1e}, x_{2e}, R_e, P_e, T_e) = 0,$ (13)

$F_3(x_{1e}, x_{2e}, R_e, P_e, T_e) = 0,$ (14)

$F_4(x_{1e}, x_{2e}, R_e, P_e, T_e) = 0,$ (15)

$F_5(x_{1e}, x_{2e}, R_e, P_e, T_e) = 0.$ (16)

The theory of stability is a well-established concept in mathematics. May (1972) defined stability as the return to equilibrium as determined by eigenvalues of the Jacobian matrix of a mathematical system. In considering the stability of the system, the interacting functions are coded in a MATLAB computer programming language in order to study the behaviour of the system, since a nonlinear system does not have a closed form solution [Baker and Rihan (1999) and Baker *et al.* (2005)].

Steady-state solution

A dynamical system is said to reach a steady-state or state of equilibrium when it exhibits no further tendency to change its property over time. That is, if the system is in a steady-state at time t_0, then it will stay there for all times $t \geq t_0$. A detailed definition and mathematical analysis of the concept of steady-state and its stability is reported in the works of Murray (2002), Ford *et al.* (2010), and Yan and Ekaka-a (2011).

Following equation (11), and if the expressions on the right hand sides are not equal to zero, then equation (1) – (5) respectively yield

$x_1 = 0$ or $x_1 = \frac{1}{a_2}(a_1 - \alpha x_2 + \alpha_1 R - k_1\delta_1 T)$. (17)

$x_2 = 0$ or $x_2 = \frac{1}{b_2}(b_1 - \beta x_1 + \beta_1 R - k_2\delta_2 T)$. (18)

$R = 0$ or $R = \frac{1}{c_2}(c_1 - \alpha_1 x_1 - \beta_1 x_2 - k\gamma T)$. (19)

$P = \frac{\eta(x_1+x_2)}{\lambda_0+\theta}$ (20)

$T = \frac{Q+\mu\theta P}{\delta_0+\delta_1 x_1+\delta_2 x_2+\gamma R}$. (21)

Hence, $\left(0, 0, 0, 0, \frac{Q}{\delta_0}\right)$ is a steady-state solution of the dynamical system (1) – (5). At this steady-state solution, the two biological species, the resource biomass and the cumulative concentration of precursors forming the toxicant will go into extinction.

When $x_1 \neq 0$, $x_2 \neq 0$, $R \neq 0$, then

$x_1 = \frac{1}{a_2}(a_1 - \alpha x_2 + \alpha_1 R - k_1\delta_1 T) = x_1^*$,

(22) $x_2 = \frac{1}{b_2}(b_1 - \beta x_1^* + \beta_1 R^* - k_2\delta_2 T^*) = x_2^*$,

(23)

$R = \frac{1}{c_2}[b_2 c_1 - b_2\alpha_1 x_1^* - \beta_1(b_1 - \beta x_1^* + \beta_1 R^* - k_2\delta_2 T^*) - b_2 k\gamma T^*] = R^*$,

(24)

$P = \frac{\eta(x_1^*+x_2^*)}{\lambda_0+\theta} = \frac{\eta[b_2 x_1^* + b_1 - \beta x_1^* + \beta_1 R^* - k_2\delta_2 T^*]}{b_2(\lambda_0+\theta)} = P^*$,

(25) $T = \frac{Q+\mu\theta P^*}{\delta_0+\delta_1 x_1^*+\delta_2 x_2^*+\gamma R^*} = \frac{b_2(Q+\mu\theta P^*)}{b_2\delta_0+b_2\delta_1 x_1^*+\delta_2(b_1-\beta x_1^*+\beta_1 R^*-k_2\delta_2 T^*)+b_2\gamma R^*} = T^*$.

(26)

Hence, $(x_1^*, x_2^*, R^*, P^*, T^*)$ is the coexistence steady-state solution.

Next, let the continuous and partially differentiable interaction functions F_1, F_2, F_3, F_4 and F_5 at an arbitrary steady-state solution $(x_{1e}, x_{2e}, R_e, P_e, T_e)$ be

$F_1(x_{1e}, x_{2e}, R_e, P_e, T_e) = a_{1e}x_{1e} - a_{2e}x_{1e}^2 - \alpha x_{1e}x_{2e} + \alpha_1 x_{1e}R_e - k_1\delta_1 x_{1e}T_e$

(27)

$F_2(x_{1e}, x_{2e}, R_e, P_e, T_e) = b_1 x_{2e} - b_2 x_{2e}^2 - \beta x_{1e}x_{2e} + \beta_1 x_{2e}R_e - k_2\delta_2 x_{2e}T_e$

(28)

$F_3(x_{1e}, x_{2e}, R_e, P_e, T_e) = c_1 R_e - c_2 R_e^2 - \alpha_1 x_{1e}R_e - \beta_1 x_{2e}R_e - k\gamma R_e T_e$

(29)

$F_4(x_{1e}, x_{2e}, R_e, P_e, T_e) = \eta x_{1e} + \eta x_{2e} - (\lambda_0 + \theta)P_e$
(30)
$F_5(x_{1e}, x_{2e}, R_e, P_e, T_e) = Q + \mu\theta P_e - \delta_0 T_e - \delta_1 x_{1e} T_e - \delta_2 x_{2e} T_e - \gamma R_e T_e$
(31)
From (27) – (31), the Jacobian matrix is obtained as

$$J_1 = \begin{bmatrix} a_{11} & -\alpha x_{1e} & \alpha_1 x_{1e} & 0 & -k_1\delta_1 x_{1e} \\ -\beta x_{2e} & a_{22} & \beta_1 x_{2e} & 0 & -k_2\delta_2 x_{2e} \\ -\alpha_1 R_e & -\beta_1 R_e & a_{33} & 0 & -k\gamma R_e \\ \eta & \eta & 0 & a_{44} & 0 \\ -\delta_1 T_e & -\delta_2 T_e & -\gamma T_e & \mu\theta & a_{55} \end{bmatrix}$$

where $a_{11} = a_1 - 2a_2 x_{1e} - \alpha x_{2e} + \alpha_1 R_e - k_1\delta_1 T_e$,
$a_{22} = b_1 - 2b_2 x_{2e} - \beta x_{1e} + \beta_1 R_e - k_2\delta_2 T_e$,
$a_{33} = c_1 - 2c_2 R_e - \alpha_1 x_{1e} - \beta_1 x_{2e} - k\gamma T_e$,
$a_{44} = -(\lambda_0 + \theta)$,
$a_{55} = -(\delta_0 + \delta_1 x_{1e} + \delta_2 x_{2e} + \gamma R_e)$.

Considering Gershgorin's theorem for estimating eigenvalues as given in Lancaster and Tismenetsky (1985), if the following inequalities hold:
$\beta x_{2e} + \alpha_1 R_e + \eta + \delta_1 T_e < a_{11}$,
$\alpha x_{1e} + \beta_1 R_e + \eta + \delta_2 T_e < a_{22}$,
$\alpha_1 x_{1e} + \beta_1 x_{2e} + \gamma T_e < a_{33}$,
$\mu\theta < a_{44}$,
$k_1\delta_1 x_{1e} + k_2\delta_2 x_{2e} + k\gamma R_e < a_{55}$,
then a_{11}, a_{22}, a_{33}, a_{44} and a_{55} are also negative, hence the steady-state solution (x_1, x_2, R, P, T) is stable, otherwise, it is unstable. T
By evaluating the Jacobian elements at the steady-state solution $\left(0, 0, 0, 0, \frac{Q}{\delta_0}\right)$, the resulting Jacobian,

$$J_2 = \begin{bmatrix} \left(a_1 - \frac{k_1\delta_1 Q}{\delta_0}\right) & 0 & 0 & 0 & 0 \\ 0 & \left(b_1 - \frac{k_2\delta_2 Q}{\delta_0}\right) & 0 & 0 & 0 \\ 0 & 0 & \left(c_1 - \frac{k\gamma Q}{\delta_0}\right) & 0 & 0 \\ \eta & \eta & 0 & -(\lambda_0 + \theta) & 0 \\ -\frac{\delta_1 Q}{\delta_0} & -\frac{\delta_1 Q}{\delta_0} & -\frac{\gamma Q}{\delta_0} & \mu\theta & -\delta_0 \end{bmatrix}$$

This matrix J_2 is a lower triangular matrix (Iheagwam, 1999). Hence, the elements on the leading diagonal, that is, $\lambda_1 = \left(a_1 - \frac{k_1\delta_1 Q}{\delta_0}\right)$, $\lambda_2 = \left(b_1 - \frac{k_2\delta_2 Q}{\delta_0}\right)$, $\lambda_3 = \left(c_1 - \frac{k\gamma Q}{\delta_0}\right)$, $\lambda_4 = -(\lambda_0 + \theta)$ and $\lambda_5 = -\delta_0$ are the eigenvalues of the system at the steady-state $\left(0, 0, 0, 0, \frac{Q}{\delta_0}\right)$.

According to linear stability analysis, the system is unstable if at least one of the eigenvalues of the Jacobian evaluated at $\left(0, 0, 0, 0, \frac{Q}{\delta_0}\right)$ has a positive real part. $\lambda_1 > 0$ if $a_1 > \frac{k_1\delta_1 Q}{\delta_0}$, $\lambda_2 > 0$ if $b_1 > \frac{k_2\delta_2 Q}{\delta_0}$, $\lambda_3 > 0$ if $c_1 > \frac{k\gamma Q}{\delta_0}$, $\lambda_4 = -(\lambda_0 + \theta)$ and $\lambda_5 = -\delta_0$. On the other hand, $\lambda_1 < 0$ if $a_1 < \frac{k_1\delta_1 Q}{\delta_0}$, $\lambda_2 < 0$ if $b_1 < \frac{k_2\delta_2 Q}{\delta_0}$, $\lambda_3 < 0$ if $c_1 < \frac{k\gamma Q}{\delta_0}$, $\lambda_4 = -(\lambda_0 + \theta)$ and $\lambda_5 = -\delta_0$. These three inequalities give conditions for the stability of this particular steady state.

Results and Discussions

To facilitate the interpretation of the mathematical analysis, the following parameter values given by Agarwal and Devi (2011) are used in the simulations for the dynamical system (1) – (5):

$a_1 = 5$, $a_2 = 0.22$, $\alpha = 0.007$, $\alpha_1 = 0.02$, $k_1 = 0.1$, $\delta_1 = 0.05$, $b_1 = 3$,

$b_2 = 0.26$, $\beta = 0.008$, $\beta_1 = 0.04$, $k_2 = 0.2$, $\delta_2 = 0.04$, $\eta = 0.5$, $\lambda_0 = 0.01$,

$\theta = 3$, $\mu = 0.2$, $\delta_0 = 7$, $\gamma = 0.3$, $c_1 = 10$, $c_2 = 0.3$, $k = 0.1$, $Q = 30$.

Table 1: Assessing the variation of the intrinsic growth rate, a_1, on the type of stability for $Q = 30$, $\delta_0 = 7$, $x_1 = x_2 = R = P = 0$ and $T = \frac{Q}{\delta_0}$, using a MATLAB ODE45 numerical scheme

Example	a_1	T	λ_1	λ_2	λ_3	λ_4	λ_5	TOS
1.	5.00	4.2857	-7	-3.01	4.9786	2.9657	9.8714	Unstable
2.	0.25	4.2857	-7	-3.01	0.2286	2.9657	9.8714	Unstable
3.	0.50	4.2857	-7	-3.01	0.4786	2.9657	9.8714	Unstable

4.	0.75	4.2857	-7	-3.01	0.7286	2.9657	9.8714	Unstable
5.	1.00	4.2857	-7	-3.01	0.9786	2.9657	9.8714	Unstable
6.	1.25	4.2857	-7	-3.01	1.2286	2.9657	9.8714	Unstable
7.	1.50	4.2857	-7	-3.01	1.4786	2.9657	9.8714	Unstable
8.	1.75	4.2857	-7	-3.01	1.7286	2.9657	9.8714	Unstable
9.	2.00	4.2857	-7	-3.01	1.9786	2.9657	9.8714	Unstable
10.	2.25	4.2857	-7	-3.01	2.2286	2.9657	9.8714	Unstable
11.	2.50	4.2857	-7	-3.01	2.4786	2.9657	9.8714	Unstable
12.	2.75	4.2857	-7	-3.01	2.7286	2.9657	9.8714	Unstable
13.	3.00	4.2857	-7	-3.01	2.9786	2.9657	9.8714	Unstable
14.	3.25	4.2857	-7	-3.01	3.2286	2.9657	9.8714	Unstable
15.	3.50	4.2857	-7	-3.01	3.4786	2.9657	9.8714	Unstable
16.	3.75	4.2857	-7	-3.01	3.7286	2.9657	9.8714	Unstable
17.	4.00	4.2857	-7	-3.01	3.9786	2.9657	9.8714	Unstable
18.	4.25	4.2857	-7	-3.01	4.2286	2.9657	9.8714	Unstable
19.	4.50	4.2857	-7	-3.01	4.4786	2.9657	9.8714	Unstable
20.	4.75	4.2857	-7	-3.01	4.7286	2.9657	9.8714	Unstable

Table 1 shows that, when $Q = 30$, for 0.25$\leq a_1 \leq 5.00$, the system has three positive and two negative eigenvalues respectively. Hence, the system is unstable and the steady-state solution is $(0, 0, 0, 0, 4.2857)$ for all values of a_1 with a constant concentration, $T = 4.2857$, of the same toxicant in the environment. However, as the value of Q is increasing, the chance of having an unstable steady-state is decreasing (as shown in Table 2).

Table 2: Assessing the variation of the intrinsic growth rate, a_1, on the type of stability for $Q = 3000, \delta_0 = 7,\ x_1 = x_2 = R = P = 0$ and $T = \frac{Q}{\delta_0}$, using a MATLAB ODE45 numerical scheme

Example	$\boldsymbol{a_1}$	T	$\boldsymbol{\lambda_1}$	$\boldsymbol{\lambda_2}$	$\boldsymbol{\lambda_3}$	$\boldsymbol{\lambda_4}$	$\boldsymbol{\lambda_5}$	TOS
1.	5.00	428.5714	-7	-3.01	2.8571	-0.4286	-2.8571	Unstable
2.	0.25	428.5714	-7	-3.01	-1.8929	-0.4286	-2.8571	Stable
3.	0.50	428.5714	-7	-3.01	-1.6429	-0.4286	-2.8571	Stable
4.	0.75	428.5714	-7	-3.01	-1.3929	-0.4286	-2.8571	Stable
5.	1.00	428.5714	-7	-3.01	-1.1429	-0.4286	-2.8571	Stable
6.	1.25	428.5714	-7	-3.01	-0.8929	-0.4286	-2.8571	Stable

7.	1.50	428.5714	-7	-3.01	-0.6429	-0.4286	-2.8571	Stable
8.	1.75	428.5714	-7	-3.01	-0.3929	-0.4286	-2.8571	Stable
9.	2.00	428.5714	-7	-3.01	-0.1429	-0.4286	-2.8571	Stable
10.	2.25	428.5714	-7	-3.01	0.1071	-0.4286	-2.8571	Unstable
11.	2.50	428.5714	-7	-3.01	0.3571	-0.4286	-2.8571	Unstable
12.	2.75	428.5714	-7	-3.01	0.6071	-0.4286	-2.8571	Unstable
13.	3.00	428.5714	-7	-3.01	0.8571	-0.4286	-2.8571	Unstable
14.	3.25	428.5714	-7	-3.01	1.1071	-0.4286	-2.8571	Unstable
15.	3.50	428.5714	-7	-3.01	1.3571	-0.4286	-2.8571	Unstable
16.	3.75	428.5714	-7	-3.01	1.6071	-0.4286	-2.8571	Unstable
17.	4.00	428.5714	-7	-3.01	1.8571	-0.4286	-2.8571	Unstable
18.	4.25	428.5714	-7	-3.01	2.1071	-0.4286	-2.8571	Unstable
19.	4.50	428.5714	-7	-3.01	2.3571	-0.4286	-2.8571	Unstable
20.	4.75	428.5714	-7	-3.01	2.6071	-0.4286	-2.8571	Unstable

Table 2 reveals that when $Q = 3000$ and $0.25 \leq a_1 \leq 2.00$, the system has all five negative eigenvalues while for $2.25 \leq a_1 \leq 5.00$, the system has four negative and one positive eigenvalues respectively. On the basis of stability theory, this steady-state solution is stable for $0.25 \leq a_1 \leq 2.00$. It can also be observed that the region of bifurcation from a stable steady-state solution to an unstable steady-state solution lies between $a_1 = 2.00$ and $a_1 = 2.25$. It is, therefore, evident that the stability of the system would depend on the cumulative rate of emission of the same toxicant into the environment from various sources. This is consistent with the result reported in Samanta and Matti (2004)

Conclusion and Recommendation

The impact of variation of the intrinsic growth rate, a_1, and the cumulative rate of emission of the same toxicants, Q, is considered. For the purpose of the study, a system of nonlinear first order ordinary differential equations was used to study the survival of a resource-dependent competing species (such as the local oil refiner and the fish farmer) when both competing species and their resources are simultaneously affected by a toxicant released into the environment from different sources. Analyses of steady-state solutions were considered and it was found that, irrespective of the variation of the intrinsic growth rate, a_1, of the first species, the stability of the system would depend on the cumulative rate of emission of the same toxicant, Q, into the environment from different sources. A further extension of this analysis can consider a second order nonlinear dynamical system.

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