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MATHEMATICAL MODELS DESCRIBING LOCAL CULT GROUP CONFLICTS

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Abstract
A linear mathematical model is proposed to study the relationship between two local cult groups, X and Y. A system of two-dimensional first order ordinary differential equation was considered in this investigation. Stability analysis of the coexistence steady-state solution of the two cult groups is incorporated and a MATLAB ODE45 numerical scheme was used in the computation. The key results of the study show that, a decrease in the ambition or grievance of a cult group discourages arms race. Also, a decrease in the efficiency of increasing the number of arms and ammunition of the cult groups, while the cost of arms and ammunition is fixed, results in a stable system. However, a decrease in the cost of arms and ammunition, while the efficiency of increasing the number of arms and ammunition is fixed, results in an unstable system. Finally, whenever the efficiency of increasing the number of arms and ammunition of the cult group is greater than the cost of arms and ammunition, the system remains unstable for all times.

***Keywords**: Stability, system of differential equation, arms and ammunition, steady-state solution and ambition or grievance.*

INTRODUCTION

Competition occurs naturally between living organisms which coexist in the same environment, especially when limited amount of resources are available. Ambition or grievances also results in competition or unhealthy rivalry between human beings. Furthermore, arms race is intense and communities have been sacked due to supremacy battles between local cult groups.

Ordinary differential equations (ODEs) is an important tool for studying the relationship between various dynamical quantities and have been applied in several fields such as population dynamics, physics, engineering and international relationships. Some researchers have provided mathematical models of conflict or combats for military or commercial purposes [Lanchester (1916); Richardson (1960a & 1960b); Morse & Kimball (1998)]. In this paper, consideration is given to the study of a mathematical model to analyse the relationship between two local cult groups.

MATERIAL AND METHODS

Suppose $x(t)$ is the armoury of a local cult group X and $y(t)$ is the armoury of a rivalry local cult group Y at time t. The rate of change of arms and ammunition on one side depends on the number of arms and ammunition on the opposing side. This is because, if one group increases its arms and ammunition and also forms alliances with other splinter cult groups, the opposing group will do likewise. That is, $\frac{dx}{dt}$ is proportional to y and $\frac{dy}{dt}$ is proportional to x. Let a_1 and b_1 be constants of proportionality to x and y respectively, representing the efficiency of increasing arms and ammunition. Hence, the system of differential equation associated with the relationship between the two local cult groups can be stated as follows:

$$\frac{dx}{dt} = a_1 y, \quad x(0) \geq 0 = x_0 \tag{1}$$

$$\frac{dy}{dt} = b_1 x, \quad y(0) \geq 0 = y_0 \tag{2}$$

Equations (1) and (2) describe the relationship between the two cult groups, each of which takes steps to defend itself against any possible attack by the other.

Rewriting (1) and (2) in D-operator form gives

$$Dx = a_1 y \quad \Rightarrow Dx - a_1 y = 0 \tag{3}$$

$$Dy = b_1 x \quad \Rightarrow b_1 x - Dy = 0 \tag{4}$$

Multiplying (3) by b_1 and operating on (4) by D yields

$$b_1 Dx - b_1 a_1 y = 0 \tag{5}$$

$$b_1 Dx - D^2 y = 0 \tag{6}$$

Solving (5) and (6) gives

$$y(t) = Ae^{t\sqrt{b_1a_1}} + Be^{-t\sqrt{b_1a_1}} \qquad (7a)$$

Putting (7a) into (4) and simplifying yields

$$x(t) = \sqrt{\frac{a_1}{b_1}}\left(Ae^{t\sqrt{b_1a_1}} - Be^{-t\sqrt{b_1a_1}}\right) \qquad (7b)$$

Applying the initial conditions, $x(0) = x_0$, $y(0) = y_0$, give

$$x_0 = \sqrt{\frac{a_1}{b_1}}(A - B) \qquad (8)$$

$$y_0 = A + B \quad \Rightarrow A = y_0 - B \qquad (9)$$

Substituting (9) into (8) yields

$$x_0 = \sqrt{\frac{a_1}{b_1}}(y_0 - 2B) \Rightarrow B = \frac{1}{2}\left(y_0 - \sqrt{\frac{a_1}{b_1}}\, x_0\right) \qquad (10)$$

Similarly, substituting (10) into (9) results in

$$A = y_0 - \frac{1}{2}\left(y_0 - \sqrt{\frac{a_1}{b_1}}x_0\right)$$

$$= \frac{1}{2}\left(y_0 + \sqrt{\frac{a_1}{b_1}}\, x_0\right) \qquad (11)$$

Hence, equation (7a) and (7b) become

$$x(t) = \sqrt{\frac{a_1}{b_1}}\left[\frac{1}{2}\left(y_0 + \sqrt{\frac{a_1}{b_1}}\, x_0\right)e^{t\sqrt{b_1a_1}} - \frac{1}{2}\left(y_0 - \sqrt{\frac{a_1}{b_1}}\, x_0\right)e^{-t\sqrt{b_1a_1}}\right],$$

$$y(t) = \sqrt{\frac{a_1}{b_1}}\left[\frac{1}{2}\left(y_0 + \sqrt{\frac{a_1}{b_1}}\, x_0\right)e^{t\sqrt{b_1a_1}} + \frac{1}{2}\left(y_0 - \sqrt{\frac{a_1}{b_1}}\, x_0\right)e^{-t\sqrt{b_1a_1}}\right] \qquad (12)$$

When y is constant, say c, it follows from (1) that

$$\frac{1}{a_1} = \frac{c}{dx/dt} \qquad \Rightarrow \frac{1}{a_1} = c\,\frac{dt}{dx} \qquad (13)$$

Solving (13) gives

$$\frac{1}{a_1}x = ct + c_1 \qquad (14)$$

Assume $x(0) = 0$, it follows from (15) that

$$c_1 = 0$$

$$\therefore \frac{1}{a_1}x = ct \quad \Rightarrow \frac{1}{a_1} = \frac{c}{x}t \qquad \text{for } x > 0.$$

When X is at the same level with Y, that is, $x = c$, then

$$\frac{1}{a_1} = t.$$

Hence, $\frac{1}{a_1}$ is the time required for the cult group X to be at the same level with Y, in terms of acquisition of arms and ammunition, provided that y is constant. Assume that the efficiency of increasing the arms and ammunition for each cult group is equal and that

$$a_1 = b_1 = 0.5$$

with the initial condition

$$x(0) = x_0 = 10, \quad y(0) = y_0 = 0.$$

It follows from (7a) and (7b) that

$$A + B = 0 \text{ and } A - B = 10.$$

Which gives

$$A = 5 \text{ and } B = -5$$

Hence, (12) simplifies to

$$x(t) = 5e^{0.5t} + 5e^{-0.5t}, \quad y(t) = 5e^{0.5t} - 5e^{-0.5t} \tag{15}$$

Equation (15) means that the arms and ammunition of each cult group increased simultaneously. Hence, it can be concluded that, when A is positive, both $x(t)$ and $y(t)$ approach infinity, which means an acceleration in the arms race, possibly leading to cult wars between the two groups, at the least provocation.

Steady-state Solution

A dynamical system is said to reach a steady-state or a state of equilibrium when it exhibits no further tendency to change its property over time (George, 2019). That is, if the system is in a steady-state at time t_0, then it will stay there for all times $t \geq t_0$. A detailed definition and mathematical analysis of the concept of steady-state and its stability is reported in the works of Halanay (1966), May & Leonard (1975), Gopalsamy (1992), Glendinning (1994), Nayfeh & Balachandran (1995), Murray (2002), Ford *et al.* (2010), and Yan & Ekaka-a (2011).

According to linear stability analysis, a steady-state solution is stable if all the eigenvalues of the Jacobian, evaluated at that steady-state solution, have negative real parts. The system is unstable if at least one of the eigenvalues has a positive real part (Ford *et al.*, 2010 and George, 2019).

At a steady-state solution all rates of change are equal to zero. Hence, equations (1) and (2) become

$$\frac{dx}{dt} = \frac{dy}{dt} = 0 \qquad \Rightarrow x = c_2 \quad \text{and} \quad y = c_3$$

This implies that, at a steady-state solution, the size of the armoury of each cult group remains constant.

Hence, it follows from (1) and (2) that

$$a_1 y = 0 \quad \text{and} \quad b_1 x = 0 \qquad \Rightarrow x = 0 \quad \text{and} \quad y = 0$$

Thus, (0, 0) is a trivial steady-state solution of the system. At this state, both cult groups have no armoury.

Characterisation of the steady-state solution

To study the behaviour of the steady-state solution, let the continuous and partially differentiable functions f and g at an arbitrary steady-state solution $(x_e\,, y_e)$ be

$$f(x_e\,, y_e) = a_1 y_e \tag{16}$$

$$g(x_e\,, y_e) = b_1 x_e \tag{17}$$

By taking the partial derivatives of f and g with respect to x_e and y_e, the following Jacobian elements are obtained:

$$J_{11} = \frac{\partial f}{\partial x_e} = 0, \qquad J_{12} = \frac{\partial f}{\partial y_e} = a_1$$

$$J_{21} = \frac{\partial g}{\partial x_e} = b_1, \qquad J_{22} = \frac{\partial g}{\partial y_e} = 0$$

giving the Jacobian matrix

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix}$$

From the characteristic equation, $|J - \lambda I| = 0$, it is given that

$$\lambda^2 - a_1 b_1 = 0 \qquad \Rightarrow \lambda_{1,2} = \pm\sqrt{a_1 b_1}$$

The eigenvalues have opposite signs, hence, the system (1) and (2) has a saddle point which is always unstable.

A Better Model

In developing a more realistic model to describe the relationship or alliances between the two cult groups, it is necessary to consider some factors such as the cost of arms and ammunition and other light weapons, and ambitions or grievances between the two groups that affect the rate of change, $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Following Ashford, (1993) and Rapoport (1957), the system describing the relationship between the two local cult groups, X and Y, is as follows:

$$\frac{dx}{dt} = a_1 y - a_2 x + \alpha, \qquad x(0) \geq 0 = x_0 \tag{18}$$

$$\frac{dy}{dt} = b_1 x - b_2 y + \beta, \qquad y(0) \geq 0 = y_0 \tag{19}$$

where a_1, a_2, b_1, b_2, α, $\beta > 0$; $x(t)$ and $y(t)$ are the arms and ammunition of X and Y; a_1 and b_1 represent the efficiencies of increasing the arms and ammunition of X and Y; a_2 and b_2 represent the cost of arms and ammunition; α and β denote the ambition or grievances that X has towards Y, and vice versa, affecting the rates of change, $\frac{dx}{dt}$ and $\frac{dy}{dt}$, positively; $-a_2 x$ and $-b_2 y$ represent the influence of the cost of arms and ammunition.

Rewriting (18) and (19), the system takes the form

$$(D + a_2)x - a_1 y = \alpha \tag{20}$$

$$b_1 x - (D + b_2)y = -\beta \tag{21}$$

$$\Rightarrow [D^2 + (a_2 + b_2)D + (a_2 b_2 - a_1 b_1)]y = a_2\beta + b_1\alpha$$

This gives a complementary function, y_c, and particular integral, y_p, as

$$y_c = Ae^{k_1 t} + Be^{k_2 t},$$

$$y_p = \frac{a_2\beta + b_1\alpha}{a_2 b_2 - a_1 b_1}.$$

Hence,

$$y(t) = Ae^{k_1 t} + Be^{k_2 t} + y_p \tag{22}$$

where

$$k_1 = -\frac{1}{2}(a_2 + b_2) + \frac{1}{2}\sqrt{(a_2 - b_2)^2 + 4a_1 b_1}$$

$$k_2 = -\frac{1}{2}(a_2 + b_2) - \frac{1}{2}\sqrt{(a_2 - b_2)^2 + 4a_1 b_1}$$

Substituting (22) into (21) gives

$$b_1 x - (D + b_2)\left[Ae^{k_1 t} + Be^{k_2 t} + y_p\right] = -\beta$$

$$\Rightarrow b_1 x - \left[Ak_1 e^{k_1 t} + Bk_2 e^{k_2 t} - Ab_2 e^{k_1 t} - Bb_2 e^{k_2 t} - b_2 y_p\right] = -\beta$$

$$\Rightarrow x(t) = \frac{1}{b_1}\left[(k_1 - b_2)Ae^{k_1 t} + (k_2 - b_2)Be^{k_2 t} - b_2 y_p\right] - \frac{\beta}{b_1}$$
(23)

Applying the initial condition, $x(0) = x_0$, $y(0) = y_0$, in (22) and (23), give

$$y_0 = A + B + y_p \quad \Rightarrow A = y_0 - y_p - B \qquad (24)$$

$$x_0 = \frac{k_1 - b_1}{b_1}A + \frac{k_2 - b_2}{b_1}B - b_2 y_p - \frac{\beta}{b_1}$$
(25)

Substituting (23) into (24) and simplifying gives

$$B = \frac{b_1 x_0 + b_1 b_2 y_p + \beta - (k_1 - b_1)(y_0 - y_p)}{k_2 - k_1 + b_1 - b_2} \qquad (26)$$

Putting (26) into (24) yields

$$A = y_0 - y_p - \left[\frac{b_1 x_0 + b_1 b_2 y_p + \beta - (k_1 - b_1)(y_0 - y_p)}{k_2 - k_1 + b_1 - b_2}\right] \qquad (27)$$

Steady-state Solution of the System

When the number of arms and ammunition is constant, that is,

$$\frac{dx}{dt} = \frac{dy}{dt} = 0,$$

it follows from (18) and (19) that

$$a_1 y - a_2 x + \alpha = 0 \Rightarrow x = \frac{1}{a_2}(a_1 y + \alpha) \qquad (28)$$

$$b_1 x - b_2 y + \beta = 0 \Rightarrow y = \frac{1}{b_2}(b_1 x + \beta) \qquad (29)$$

Substituting (29) into (28) and vice versa yield

$$x = \frac{a_1 \beta + b_2 \alpha}{a_2 b_2 - a_1 b_1} = x_1, \; y = \frac{b_1 \alpha + a_2 \beta}{a_2 b_2 - a_1 b_1} = y_1$$

(x_1, y_1) is the only steady-state solution of system (18) and (19).

Characterisation of the Steady-state Solution

Let the continuous and partially differentiable functions f and g at an arbitrary steady-state solution (x_e, y_e) be

$$f(x_e, y_e) = a_1 y_e - a_2 x_e + \alpha \qquad (29)$$

$$g(x_e, y_e) = b_1 x_e - b_2 y_e + \beta \qquad (30)$$

Taking the partial derivatives of f and g with respect to x_e and y_e, give the following Jacobian elements:

$$J_{11} = \frac{\partial f}{\partial x_e} = -a_2, \quad J_{12} = \frac{\partial f}{\partial y_e} = a_1$$

$$J_{21} = \frac{\partial g}{\partial x_e} = b_1, \qquad J_{22} = \frac{\partial g}{\partial y_e} = -b_2$$

Hence, the Jacobian matrix is

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} -a_2 & a_1 \\ b_1 & -b_2 \end{bmatrix}$$

The characteristic equation, $|J - \lambda I| = 0$, results in

$$\begin{vmatrix} -a_2 - \lambda & a_1 \\ b_1 & -b_2 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + (a_2 + b_2)\lambda + (a_2 b_2 - a_1 b_1) = 0$$

$$\Rightarrow \lambda_{1,2} = -\frac{1}{2}(a_2 + b_2) \pm \frac{1}{2}\sqrt{(a_2 + b_2)^2 - 4(a_2 b_2 - a_1 b_1)}$$

$$= -\frac{1}{2}(a_2 + b_2) \pm \frac{1}{2}\sqrt{(a_2 - b_2)^2 + 4a_1 b_1}$$

RESULTS AND DISCUSSIONS

In the numerical simulation for the dynamical system (18) and (19), the following parameter values are used: $a_1 = 0.35$, $a_2 = 0.3$, $b_1 = 0.4$, $b_2 = 0.5$, $\alpha = 0.7$ and $\beta = 0.9$

Table 1: Effect of variation of ambition or grievance of cult group X, towards cult group Y, on the quantity of arms and ammunition of X and Y, using a MATLAB ODE45 numerical scheme

S/N	α	β	x	y
1	0.7000	0.9000	15.2964	12.9694
2	0.6650	0.9000	14.8910	12.6731
3	0.6300	0.9000	14.4859	12.3769
4	0.5950	0.9000	14.0802	12.0806
5	0.5600	0.9000	13.6749	11.7843
6	0.5250	0.9000	13.2695	11.4881
7	0.4900	0.9000	12.8641	11.1918
8	0.4550	0.9000	12.4588	10.8956
9	0.4200	0.9000	12.0534	10.5993
10	0.3850	0.9000	11.6480	10.3030
11	0.3500	0.9000	11.2426	10.0068
12	0.3150	0.9000	10.8373	9.7105
13	0.2800	0.9000	10.4319	9.4143

14	0.2450	0.9000	10.0265	9.1180
15	0.2100	0.9000	9.6211	8.8217
16	0.1750	0.9000	9.2158	8.5255
17	0.1400	0.9000	8.8104	8.2292
18	0.1050	0.9000	8.4050	7.9329
19	0.0700	0.9000	7.9996	7.6367
20	0.0350	0.9000	7.5943	7.3404

Table 2: Effect of variation of ambition or grievance of cult group Y, towards cult group X, on the quantity of arms and ammunition of x and y, using a MATLAB ODE45 numerical scheme

S/N	α	β	x	y
1	0.7000	0.9000	15.2964	12.9694
2	0.7000	0.8550	14.9631	12.6386
3	0.7000	0.8100	14.6298	12.3079
4	0.7000	0.7650	14.2965	11.9772
5	0.7000	0.7200	13.9632	11.6464
6	0.7000	0.6750	13.6299	11.3157
7	0.7000	0.6300	13.2966	10.9849
8	0.7000	0.5850	12.9633	10.6542
9	0.7000	0.5400	12.6300	10.3234
10	0.7000	0.4950	12.2967	9.9927
11	0.7000	0.4500	11.9634	9.6620
12	0.7000	0.4050	11.6301	9.3312
13	0.7000	0.3600	11.2969	9.0005
14	0.7000	0.3150	10.9636	8.6697
15	0.7000	0.2700	10.6303	8.3390
16	0.7000	0.2250	10.2970	8.0082
17	0.7000	0.1800	9.9637	7.6775
18	0.7000	0.1350	9.6304	7.3468
19	0.7000	0.0900	9.2971	7.0160
20	0.7000	0.0400	8.9638	6.6853

Table 3a: Effect of variation of the efficiency of increasing the number of arms and ammunition, a_1b_1, of cult groups, X and Y, on the stability of the system, using a MATLAB ODE45 numerical scheme

S/N	a_1	a_1b_1	a_2b_2	λ_1	λ_2	TOS
1	0.3500	0.1400	0.1500	-0.0127	-0.7873	Stable
2	0.3325	0.1330	0.1500	-0.0218	-0.7782	Stable
3	0.3150	0.1260	0.1500	-0.0312	-0.7688	Stable
4	0.2975	0.1190	0.1500	-0.0408	-0.7595	Stable
5	0.2800	0.1120	0.1500	-0.0507	-0.7493	Stable
6	0.2625	0.1050	0.1500	-0.0609	-0.7391	Stable
7	0.2450	0.0980	0.1500	-0.0714	-0.7286	Stable
8	0.2275	0.0910	0.1500	-0.0822	-0.7178	Stable
9	0.2100	0.0840	0.1500	-0.0934	-0.7066	Stable
10	0.1925	0.0770	0.1500	-0.1050	-0.6950	Stable
11	0.1750	0.0700	0.1500	-0.1172	-0.6828	Stable
12	0.1575	0.0630	0.1500	-0.1298	-0.6707	Stable
13	0.1400	0.0560	0.1500	-0.1431	-0.6569	Stable
14	0.1225	0.0490	0.1500	-0.1571	-0.6429	Stable
15	0.1050	0.0420	0.1500	-0.1720	-0.6280	Stable
16	0.0875	0.0350	0.1500	-0.1879	-0.6121	Stable
17	0.0700	0.0280	0.1500	-0.2051	-0.5949	Stable
18	0.0525	0.0210	0.1500	-0.2239	-0.5761	Stable
19	0.0350	0.0140	0.1500	-0.2451	-0.5549	Stable
20	0.0175	0.0070	0.1500	-0.2696	-0.5304	Stable

Table 3b: Effect of variation of the efficiency of increasing the number of arms and ammunition, a_1b_1, of cult groups, X and Y, on the stability of the system, using a MATLAB ODE45 numerical scheme

S/N	a_1	b_1	a_1b_1	a_2b_2	λ_1	λ_2	TOS
1	0.3500	0.4000	0.1400	0.1500	-0.0127	-0.7873	Stable
2	0.3325	0.3800	0.1263	0.1500	-0.0307	-0.7693	Stable
3	0.3150	0.3600	0.1134	0.1500	-0.0487	-0.7513	Stable
4	0.2975	0.3400	0.1012	0.1500	-0.0666	-0.7334	Stable
5	0.2800	0.3200	0.0896	0.1500	-0.0844	-0.7156	Stable

6	0.2625	0.3000	0.0788	0.1500	-0.1021	-0.6979	Stable
7	0.2450	0.2800	0.0686	0.1500	-0.1196	-0.6804	Stable
8	0.2275	0.2600	0.0591	0.1500	-0.1370	-0.6630	Stable
9	0.2100	0.2400	0.0504	0.1500	-0.1542	-0.6458	Stable
10	0.1925	0.2200	0.0424	0.1500	-0.1712	-0.6288	Stable
11	0.1750	0.2000	0.0350	0.1500	-0.1879	-0.6121	Stable
12	0.1575	0.1800	0.0284	0.1500	-0.2042	-0.5958	Stable
13	0.1400	0.1600	0.0224	0.1500	-0.2200	-0.5800	Stable
14	0.1225	0.1400	0.0171	0.1500	-0.2352	-0.5648	Stable
15	0.1050	0.1200	0.0126	0.1500	-0.2497	-0.5503	Stable
16	0.0875	0.1000	0.0087	0.1500	-0.2631	-0.5369	Stable
17	0.0700	0.0800	0.0056	0.1500	-0.2751	-0.5249	Stable
18	0.0525	0.0600	0.0031	0.1500	-0.2853	-0.5147	Stable
19	0.0350	0.0400	0.0014	0.1500	-0.2932	-0.5068	Stable
20	0.0175	0.0200	0.0004	0.1500	-0.2983	-0.5017	Stable

Table 4a: Effect of variation of the cost, $a_2 b_2$, of arms and ammunition, on the stability of the system, using a MATLAB ODE45 numerical scheme

S/N	a_2	$a_1 b_1$	$a_2 b_2$	λ_1	λ_2	TOS
1	0.3000	0.1400	0.1500	-0.0127	-0.7873	Stable
2	0.2850	0.1400	0.1425	-0.0032	-0.7818	Stable
3	0.2700	0.1400	0.1350	0.0064	-0.7764	Unstable
4	0.2550	0.1400	0.1275	0.0162	-0.7712	Unstable
5	0.2400	0.1400	0.1200	0.0261	-0.7661	Unstable
6	0.2250	0.1400	0.1125	0.0361	-0.7611	Unstable
7	0.2100	0.1400	0.1050	0.0463	-0.7563	Unstable
8	0.1950	0.1400	0.0975	0.0565	-0.7515	Unstable
9	0.1800	0.1400	0.0900	0.0669	-0.7469	Unstable
10	0.1650	0.1400	0.0825	0.0774	-0.7424	Unstable
11	0.1500	0.1400	0.0750	0.0881	-0.7381	Unstable
12	0.1350	0.1400	0.0675	0.0988	-0.7338	Unstable
13	0.1200	0.1400	0.0600	0.1096	-0.7296	Unstable
14	0.1050	0.1400	0.0525	0.1206	-0.7256	Unstable

15	0.0900	0.1400	0.0450	0.1316	-0.7216	Unstable
16	0.0750	0.1400	0.0375	0.1428	-0.7178	Unstable
17	0.0600	0.1400	0.0300	0.1541	-0.7141	Unstable
18	0.0450	0.1400	0.0225	0.1654	-0.7104	Unstable
19	0.0300	0.1400	0.0150	0.1768	-0.7068	Unstable
20	0.0150	0.1400	0.0075	0.1884	-0.7034	Unstable

Table 4b: Effect of variation of the cost, a_2b_2, of arms and ammunition, on the stability of the system, using a MATLAB ODE45 numerical scheme

S/N	a_2	b_2	a_1b_1	a_2b_2	λ_1	λ_2	TOS
1	0.3000	0.5000	0.1400	0.1500	-0.0127	-0.7873	Stable
2	0.2850	0.4750	0.1400	0.1354	0.0060	-0.7660	Unstable
3	0.2700	0.4500	0.1400	0.1215	0.0248	-0.7448	Unstable
4	0.2550	0.4250	0.1400	0.1084	0.0437	-0.7237	Unstable
5	0.2400	0.4000	0.1400	0.0960	0.0626	-0.7026	Unstable
6	0.2250	0.3750	0.1400	0.0844	0.0816	-0.6816	Unstable
7	0.2100	0.3500	0.1400	0.0735	0.1007	-0.6607	Unstable
8	0.1950	0.3250	0.1400	0.0634	0.1198	-0.6398	Unstable
9	0.1800	0.3000	0.1400	0.0540	0.1389	-0.6189	Unstable
10	0.1650	0.2750	0.1400	0.0454	0.1582	-0.5982	Unstable
11	0.1500	0.2500	0.1400	0.0375	0.1775	-0.5775	Unstable
12	0.1350	0.2250	0.1400	0.0304	0.1969	-0.5569	Unstable
13	0.1200	0.2000	0.1400	0.0240	0.2163	-0.5363	Unstable
14	0.1050	0.1750	0.1400	0.0184	0.2358	-0.5158	Unstable
15	0.0900	0.1500	0.1400	0.0135	0.2554	-0.4954	Unstable
16	0.0750	0.1250	0.1400	0.0094	0.2750	-0.4750	Unstable
17	0.0600	0.1000	0.1400	0.0060	0.2947	-0.4547	Unstable
18	0.0450	0.0750	0.1400	0.0034	0.3145	-0.4345	Unstable
19	0.0300	0.0500	0.1400	0.0015	0.3343	-0.4143	Unstable
20	0.0150	0.0250	0.1400	0.0004	0.3542	-0.3942	Unstable

Table 1 shows that a decrease in the ambition or grievances, α, of the cult group, X, while the ambition or grievance, β, of the cult group, Y, is fixed, results in a decrease in the quantity of arms and ammunition of both groups. Similarly,

Table 2 shows that a decrease in the ambition or grievance, β, of the cult group, Y, while the ambition or grievances, α, of the cult group, X, is fixed, results in a decrease in the quantity of arms and ammunition of both groups.

Tables 3a and 3b reveal that a decrease in the efficiency of increasing the number of arms and ammunition, a_1b_1, of cult groups X and Y, while the cost of arms and ammunition, a_2b_2, is fixed, results in a stable system. In contrast, Tables 4a and 4b have shown that a decrease in the combined cost of arms and ammunition, a_2b_2, while the combined efficiency of increasing the number of arms and ammunition, a_1b_1, of cult groups X and Y, is fixed, results in an unstable system. It can also be stated that whenever $a_1b_1 > a_2b_2$, that is, for as long as the combined efficiency of increasing the number, a_1b_1, of arms and ammunition, of both cult groups is greater than the combined cost, a_2b_2, of arms and ammunition, the system remains unstable.

CONCLUSION AND RECOMMENDATION

A linear mathematical model was proposed to study the relationship between two cult groups, X and Y, and the stability analysis of the coexistence steady-state solution is incorporated. A MATLAB ODE45 numerical scheme was used in the computation. The key contribution of the investigation shows that:

(i) a decrease in the ambition or grievance of a cult group discourages arms race between the groups.

(ii) a decrease in the efficiency of increasing the number of arms and ammunition of the cult groups, while the cost of arms and ammunition is fixed, results in a stable system.

(iii) a decrease in the cost of arms and ammunition, while the efficiency of increasing the number of arms and ammunition is fixed, results in an unstable system.

(iv) whenever the efficiency of increasing the number of arms and ammunition of the cult group is greater than the cost of arms and ammunition, the system remains unstable for all times.

A system of two-dimensional ordinary differential equation was considered in this investigation. For further investigation, a linear system of three-dimensional ordinary differential equation is recommended to study the relationship between the two cult groups, considering the effect of government policies.

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