GRONWALL'S INEQUALITY FOR INTEGRO-DIFFERENTIAL DELAY EQUATIONS AND ITS APPLICATIONS

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Abstract
The paper analyses the asymptotic behaviour of differential equations when subject to a delay system with basic example. The scheme developed give a favourable solution that will be useful to any system under consireation.

Key words: Delay equations, integro-differential equations, gronwall's inequalities.

Introduction

In many real-world systems, like science, industry, economics and finance etc., time delay is what we will encounter. The differential delay equations (DDEs) including the functional differential equations have been used to model such time-delay systems. Since the time-delay often 5 causes the instability of systems, stability of DDEs has been researched intensively for more than 50 years

It is a known fact; there are existing classes of mathematical models described by differential equations, like Malthus' population model. There are many differential equations that do not possess the exact solution. In such situation, integral inequalities are significant for investigating the boundedness, stability, asymptotic behavior of solutions to differential equations. In 1919, Gronwall put forward what is now well-known Gronwall's inequality to estimate the solution of linear differential equation. Whilst Bihari(1956) developed Bihari's inequality by extending Gronwall's inequality to nonlinear one; there are many authors who have been devoted to studying New non-linear Integral Inequalities(NII) in recent years. Example, based on the universal Gronwall's inequality, Tian et al.(2015) investigated the asymptotic behavior of switched delay systems that represent a class of systems in practical engineering and have wide application in automated highways, power systems. Tian and Fan (2020) studied the application of nonlinear integral inequalities in a delayed integro-differential equation and a favourable result.

Gronwall Type Inequalities for the System: $x(t) = L(x, t)$ (1)

Let r be a continuous non-negative function on an interval J = [a, b] ⊂ R and S, K are non-negative constant such that:

$r(t) = \delta + \int_a^b Kr(s)ds, \qquad \forall t \in J$ (2)

Then:

$$r(t) = \delta exp\{k(t-a)\}, \qquad \forall t \in J$$

proof:

Define $R(t) = \delta + \int_a^b Kr(s)ds$ (3)

By differentiating, gives:

$R^1(t)=Kr(t)$ (4)

⇨ $R^1(t) - Kr(t)=0$ (5)

And also, $R(a) = \delta,\ for\ R(t) \geq r(t),$ given

$\therefore\ R'(t) - KR(t) \leq R'(t) - Kr(t) = 0$ (6)

=> $\int \frac{R^1(t)}{R(t)} dt = \int_a^t Kdt => R(t) = e^{k(t-a)}$ (7)

By (3) - (5), set V(t) = exp[k(t-a)] (8)

Multiplying (6) by V(t), and obtained

$V(t)R^1(t) - Kr(t)R(t) \leq O$ or $[V(t)R(t)] \leq O$ (9)

Since, the integral from a t_o t of this non-negative function is again non-negative then integrate and obtain

[V(s)R(s)] = V(t)R(t) – V(a)R(a) (10)

$=V(t)R(t) - \delta$ (by (5)) (11)

$\therefore \quad V(t)R(t) - \delta \leq 0$, and hence

$R(t) \leq \frac{\delta}{V(t)}$ $or \ since, r(t) \leq R(t)$, gives

$$r(t) \leq R(t) \leq \frac{\delta}{V(t)} \tag{12}$$

Which is also given by; r(t) $\leq \delta \exp\,[k(t-a)]$ (13)

Lemma 1:

Let g be a continuous non-negative function such that g(t) =o, for t<t_o, and

g(t) $\leq a\int_{t_o}^{t} g(\ <)ds + b\ \int_{t_o}^{t} g(s-1)\,ds + c\ \int_{t_o}^{t}\int_{f_o}^{s} g(w)dwds +\ d$ (14)

for t$\in$[t_o,t], where a,b,c,d are non-negative constants, then:

$g(t) \leq d(m+1)K_1^m \exp\,((m+1)k_2(t_{1\_}t_o))$

(15)

where k_1 = max{b,1}, k_2 = max(a+1),c} for t$\in [t_{o,} t]$, and m is the non-negative integer such that t_o + m$\leq m \leq t_1 < t_o + m + 1$.

Proof:

Let t$\in$[t_o,t_o+1], then

g(t) $\leq a\ \int_{t_o}^{t} g(s)ds + c\ \int_{t_o}^{t}\int_{t_o}^{s} g(w)dwds + d$ (14)

Hence,

g(t)+ $\int_{t_o}^{t} g\,(w)dw\ \leq K_2\int_{t_o}^{t}[g(s) + \int_{t_o}^{s} g(w)d]ds + d$ (16)

By Gronwall's inequality,

g(t) + $\int_{t_o}^{t} g(w)d\ <\leq d\ e\ < p\,[K_2(t_1 - t_o$)] (17)

Hence, $g(t) \leq dexp[k_2 t_1 - t_o)]$ (18)

for t$\in$[t_o,t_o+1].

Similarly, if $t \in [t_o+1, t_r+2]$, then

$$g(t) \leq a\int_{t_o}^{t} g(s)ds + b\int_{t_{o+1}}^{t} d\exp[K_2(t_1 - t_o)]\,ds + C\int_{t_o}^{t}\int_{t_o}^{s} g(w)dwds + d$$

(19)

$$\leq a\int_{t_o}^{t} g(s)d\;\langle + C\int_{t}^{t} + \int_{t_o}^{t} g(w) < wds + d[bexp(K_2(t_1 - t_o)) + 1]$$

(21)

Considering again the Gronwall's inequality, one will get:

$$g(t) + \int_{t_o}^{t} g\,(w)dw \leq 2dK_1\exp(K_2(t_1 - t_1))\exp\,(K_2(t_1 - t_o))$$

(22)

Hence,

$$g(t) \leq 2dK_1\exp\,(2K_2(t_1 - t_o))$$

(23)

for $t \in [t_o+1, t_o+2]$. Continuation in the same manner yield;

$$g(t) \leq (j+1)dk\{\exp[(j+i)k_2(t_1-t_o)]\}$$

(24)

for $t \in [t_{o+j}, t_{o+j+1}]$ and j=0,1,2,…,m. Since $1 \leq k_2$, this gives:

$$g(t) \leq (m+1)dk_1^m\exp\{(m+1)k_2(t_2-t_o)\}]$$

(25)

Therefore, the Lemma stated above has been proved

Conclusion

Gronwall's inequality is a useful tool in solving real life delay systems. Its applications are favourable when dealing with power of integro-differential equations

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