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NUMERICAL SIMULATION OF THE TYPE OF STABILITY USING A MATLAB ALGORITHM

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ABSTRACT
Having studied the effects of varying the intrinsic growth rates together on the type of stability in the context of two competing technologies to obtain a dominant occurrence of stability with a zero bifurcation, we have applied the similar method of a Matlab Algorithm to obtain a total of six (6) instances of degenerate steady state solutions and forty eight (48) scenarios of valid stability. The bifurcation from a stable steady state solution to a degenerate steady state has occurred for the following pairs of the intra – competition coefficient (0.00240, 0.00360) and (0.00120, 0.00018). Next, we have found another bifurcation from a degenerate steady state solution to a stable steady state solution between examples 7 and 8 as displayed on Table 1; the novel results which we have found that we have not seen elsewhere are presented and discussed.

KEYWORDS: *Numerical simulation, Degenerate, Stability, Eigenvalues, Intra – competition coefficient, Steady state solution, Stable, Jacobian matrix, Competing technologies, Bifurcation, Interacting, Matlab Algorithm.*

INTRODUCTION
In their work, Nafo and Ekaka-a (2015) established that the variations of model parameter values that produced stability could be significantly

needed in some aspect of sustainable development and investment planning while the variations of model parameter values that produced degeneracy properties or features could be needful to avoid stock exchange crash in the stock exchange system which is mainly motivated or driven by economic interest (gain).

Following Mamat, Mada Sanjaya, Salleh and Ahmad (2011), numerical verification of any results gotten through analytical studies makes the analysis complete.

The principle of carrying capacity has to do with the highest population which can sustain and strengthen the growth of a population. The values of eigenvalues can be utilized to find the type of stability of the coexistence steady state solution by using the theory of the sign method on the stability of a steady state solution of any growth patterns.

Managers and business owners believe that growth and stability are two (2) basic things which are necessary to ensure a durable success of any business. Maximum parsimony is the principle that most acceptable explanation of an occurrence, phenomenon or event is the simplest (involving the fewest entities, assumptions, or changes).

In their research, Ekaka-a and Galadima (2015), established that the method of numerical simulation can be use to quantify the impact of the changes in the carrying capacities for two mutualistic interacting legumes and its type of stability.

The stability analysis of a mathematical model of two interacting technologies is a challenging research problem because the implication of more instances of stability is a signal for an improve business environment and trading in the context of two competing technologies. The benefit of marketing management is that this technique divides total demand into relatively similar parts which are identified by some common characteristics or features. We try to show models of a real system and conducted experiments with these models for the purpose of understanding the behavior (within the limits imposed by a criterion or set of criteria) for the growth patterns. We utilized the importance of a numerical method as a method that is designed for constructive solution of mathematical results usually on a computer. It is a mathematical tool designed to solve numerical problems (Iyalla, 2018).

SIMPLIFYING ASSUMPTIONS

For the purpose of this study, we shall consider the following assumptions (Ekaka – a, 2009) and (Zhang, McAdams, Shankarl and Mohammadi Darani, 2018):

1) The growth of the two technologies over time is directly proportional to the size of the technology, thereby making the growth rate to be called an enhancing growth rate.
2) The growth of the two technologies is inhibited by their intra-competition coefficients.
3) The growth of the two technologies is enhanced by their inter-competition coefficients.

MATHEMATICAL FORMULATIONS

In this study, we shall consider the following systems of first order differential (ODE) equations
(Ekaka – a, 2009) and (Zhang, McAdams, Shankarl and Mohammadi Darani, 2018):

I) $$\frac{dT_1}{dt} = \alpha_1 T_1 - \beta_1 T_1^{\,2} \qquad (1)$$

II) $$\frac{dT_2}{dt} = \alpha_2 T_2 - \beta_2 T_2^{\,2} \qquad (2)$$

III) $$\frac{dT_1}{dt} = \alpha_1 T_1 - \beta_1 T_1^{\,2} + r_1 T_1 T_2 \qquad (3)$$

IV) $$\frac{dT_2}{dt} = \alpha_2 T_2 - \beta_2 T_2^{\,2} + r_2 T_1 T_2 \qquad (4)$$

With initial conditions $T_1(t) = T_1(0) > 0$ and $T_2(t) = T_2(0) > 0$ at time $t = 0$ where,
$T_1(t)$ represents the size of first technology at time t.
$T_2(t)$ represents the size of second technology at time t.
α_1 and α_2 represent the intrinsic growth rate of the first technology and the second technology respectively.

$\beta_1$and β_2 represent the intra – competition coefficient for the first technology and the second technology respectively.
$r_1$and r_2 represent the inter – coefficient that enhanced the growth of the two mutualistic interacting technologies respectively.
For the purpose of this, we have assumed the following parameter values:
$\alpha_1 = 0.1445$, $\alpha_2 = 0.097$, $\beta_1 = 0.0036$, $\beta_2 = 0.0024$, $r_1 = 0.0012$, $r_2 = 0.0008$, $T_1(0) = 40$ and $T_2(0) = 40$

METHOD OF ANALYSIS

Steady State Solutions for the Lotka Inter - Competition Equations.
We consider the inter - competition equations

$$\frac{dT_1}{dt} = \alpha_1 T_1 - \beta_1 {T_1}^2 + r_1 T_1 T_2 \quad (3)$$

$$\frac{dT_2}{dt} = \alpha_2 T_2 - \beta_2 {T_2}^2 + r_2 T_1 T_2 \quad (4)$$

With initial conditions
$T_1(t) = T_1(0) > 0$ and $T_2(t) = T_2(0) > 0$ at time $t = 0$
How do we find the steady state solutions?
Following the work of Ekaka'a (2009),
Let $\frac{dT_1}{dt} = 0$ and $\frac{dT_2}{dt} = 0$ and we find the values of T_1 and T_2.These values of T_1 and
T_2 are called the steady state solutions.
At a steady state solutions, following Ekaka-a (2009), $\frac{dT_1}{dt} = 0$ and $\frac{dT_2}{dt} = 0$
Consider the non- linear first order ordinary differential equations:

$$\frac{dT_1}{dt} = \alpha_1 T_1 - \beta_1 {T_1}^2 + r_1 T_1 T_2$$

$$\frac{dT_2}{dt} = \alpha_2 T_2 - \beta_2 {T_2}^2 + r_1 T_1 T_2$$

With initial conditions $T_1(t) = T_1(0) > 0$ and $T_2(t) = T_2(0) > 0$ at time $t = 0$
Using the fact that $\frac{dT_1}{dt} = 0$
It implies that

$$\alpha_1 T_1 - \beta_1 {T_1}^2 + r_1 T_1 T_2 = 0$$

$$T_1(\alpha_1 - \beta_1 T_1 + r_1 T_2) = 0$$

Where $T_1 \neq 0$

$$\alpha_1 - \beta_1 T_1 + r_1 T_2 = 0$$
$$\beta_1 T_1 = \alpha_1 + r_1 T_2$$
$$T_1 = \frac{\alpha_1 + r_1 T_2}{\beta_1} \qquad (5)$$

Therefore, we can say that when $T_2 = 0, \quad T_1 = \frac{\alpha_1 + r_1(0)}{\beta_1} = \frac{\alpha_1 + 0}{\beta_1} = \frac{\alpha_1}{\beta_1}$

Similarly,

Using the fact that $\frac{dT_2}{dt} = 0$

It implies that

$$\alpha_2 T_2 - \beta_2 T_2^{\,2} + r_2 T_1 T_2 = 0$$
$$T_2(\alpha_2 - \beta_2 T_2 + r_2 T_1) = 0$$

Where $T_2 \neq 0$

$$\alpha_2 - \beta_2 T_2 + r_2 T_1 = 0$$
$$\beta_2 T_2 = \alpha_2 + r_2 T_1$$
$$T_2 = \frac{\alpha_2 + r_2 T_1}{\beta_2} \qquad (6)$$

Therefore, when $T_1 = 0, \quad T_2 = \frac{\alpha_2 + r_2(0)}{\beta_2} = \frac{\alpha_2 + 0}{\beta_2} = \frac{\alpha_2}{\beta_2}$

$\therefore \quad T_1 = \frac{\alpha_1 + r_1 T_2}{\beta_1}$ And $T_2 = \frac{\alpha_2 + r_2 T_1}{\beta_2}$

Assuming that $\alpha_1 - \beta_1 T_1 + r_1 T_2 \neq 0$ on the basis of algebra $T_1 = 0$

In the same way, assuming that $\alpha_2 - \beta_2 T_2 + r_2 T_1 \neq 0$ on the basis of algebra $T_2 = 0$

Solving equation (5) and (6) simultaneously by substitution method, we get

$$T_1 = \frac{\alpha_1 + r_1\left(\frac{\alpha_2 + r_2 T_1}{\beta_2}\right)}{\beta_1} = \frac{\alpha_1 \beta_2 + r_1(\alpha_2 + r_2 T_1)}{\beta_1 \beta_2}$$
$$T_1 = \frac{\alpha_1 \beta_2 + r_1(\alpha_2 + r_2 T_1)}{\beta_1 \beta_2} = \frac{\alpha_1 \beta_2 + r_1 \alpha_2 + r_1 r_2 T_1}{\beta_1 \beta_2}$$
$$T_1 = \frac{\alpha_1 \beta_2 + r_1 \alpha_2 + r_1 r_2 T_1}{\beta_1 \beta_2}$$
$$T_1 \beta_1 \beta_2 = \alpha_1 \beta_2 + r_1 \alpha_2 + r_1 r_2 T_1$$
$$T_1 \beta_1 \beta_2 - r_1 r_2 T_1 = \alpha_1 \beta_2 + r_1 \alpha_2$$
$$T_1(\beta_1 \beta_2 - r_1 r_2) = \alpha_1 \beta_2 + r_1 \alpha_2$$
$$T_1 = \frac{\alpha_1 \beta_2 + r_1 \alpha_2}{\beta_1 \beta_2 - r_1 r_2} \qquad (7)$$

Similarly,

$$T_2 = \frac{\alpha_1 + r_2\left(\frac{\alpha_1 + r_1 T_2}{\beta_2}\right)}{\beta_1} = \frac{\alpha_2\beta_1 + r_2(\alpha_1 + r_1T_2)}{\beta_1\beta_2}$$

$$T_2 = \frac{\alpha_2\beta_1 + r_2(\alpha_1 + r_1T_2)}{\beta_1\beta_2} = \frac{\alpha_2\beta_1 + r_2\alpha_1 + r_1r_2T_2}{\beta_1\beta_2}$$

$$T_2 = \frac{\alpha_2\beta_1 + r_2\alpha_1 + r_1r_2T_2}{\beta_1\beta_2}$$

$$T_2\beta_1\beta_2 = \alpha_2\beta_1 + r_2\alpha_1 + r_1r_2T_2$$

$$T_2\beta_1\beta_2 - r_1r_2T_2 = \alpha_2\beta_1 + r_2\alpha_1$$

$$T_2(\beta_1\beta_2 - r_1r_2) = \alpha_2\beta_1 + r_2\alpha_1$$

$$T_2 = \frac{\alpha_2\beta_1 + r_2\alpha_1}{\beta_1\beta_2 - r_1r_2} \qquad (8)$$

Therefore, we have obtained the following steady state solutions namely:

1. The point $(T_1, T_2) = (0, 0)$
2. The point $(T_1, T_2) = \left(\frac{\alpha_1}{\beta_1}, 0\right)$
3. The point $(T_1, T_2) = \left(0, \frac{\alpha_2}{\beta_2}\right)$
4. The point $(T_1, T_2) = \left(\frac{\alpha_1\beta_2 + r_1\alpha_2}{\beta_1\beta_2 - r_1r_2}, \frac{\alpha_2\beta_1 + r_2\alpha_1}{\beta_1\beta_2 - r_1r_2}\right)$

Linearization for the inter – competition model

Following Ekaka'a (2009), we assumed that two interacting functions of two dependent variables T_1 and T_2 exist which are continuous and have partial derivatives of first order. These interacting functions are defined as follows:

$$F(T_1, T_2) = \alpha_1T_1 - \beta_1T_1^2 + r_1T_1T_2 \qquad (9)$$

$$G(T_1, T_2) = \alpha_2T_2 - \beta_2T_2^2 + r_2T_1T_2 \qquad (10)$$

By differentiating equations (9) and (10) with respect to T_1 & T_2 respectively, we get

Let $J_{11} = \left.\frac{\partial F}{\partial T_1}\right|_T = \alpha_1 - 2\beta_1T_1 + r_1T_2$

$$J_{12} = \left.\frac{\partial F}{\partial T_2}\right|_T = r_1T_1$$

$$J_{21} = \left.\frac{\partial G}{\partial T_1}\right|_T = r_2T_2$$

$$J_{22} = \frac{\partial G}{\partial T_2}\Big|_T = \alpha_2 - 2\beta_2 T_2 + r_2 T_1$$

It implies that

$$J_{11} = \alpha_1 - 2\beta_1 T_1 + r_1 T_2$$
$$J_{12} = r_1 T_2$$
$$J_{21} = r_2 T_1$$
$$J_{22} = \alpha_2 - 2\beta_2 T_2 + r_2 T_1$$

Evaluating the Jacobian matrix's elements J_{11}, J_{12}, J_{21} and J_{22} at (1) $(T_1, T_2) = (0, 0)$

$$T_1 = 0, \ T_2 = 0$$
$$J_{11} = \alpha_1 - 2\beta_1(0) + r_1(0) = \alpha_1 - 0 + 0 = \alpha_1$$
$$J_{12} = r_1(0) = 0$$
$$J_{21} = r_2(0) = 0$$
$$J_{22} = \alpha_2 - 2\beta_2(0) + r_1(0) = \alpha_2 - 0 + 0 = \alpha_2$$

Let the Jacobian matrix at $(T_1, T_2) = (0, 0)$ be J

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$
$$J = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$$

We use the characteristics equation $|J - \lambda I| = 0$ to find the two eigenvalues.

Let $I = \lambda I$, where I is the identity matrix.

$$\therefore \quad \lambda I = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$
$$\therefore \quad J - \lambda I = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \alpha_1 - \lambda & 0 \\ 0 & \alpha_2 - \lambda \end{pmatrix}$$
$$\therefore \quad |J - \lambda I| = \begin{vmatrix} \alpha_1 - \lambda & 0 \\ 0 & \alpha_2 - \lambda \end{vmatrix}$$
$$= (\alpha_1 - \lambda)(\alpha_2 - \lambda) - (0)(0) = (\alpha_1 - \lambda)(\alpha_2 - \lambda)$$

But $|J - \lambda I| = 0$

$$\therefore \quad (\alpha_1 - \lambda)(\alpha_2 - \lambda) = 0$$

It implies that $(\alpha_1 - \lambda) = 0$ or $(\alpha_2 - \lambda) = 0$

Therefore $\lambda = \alpha_1$ or $\lambda = \alpha_2$

Where $\alpha_1 = 0.0970$ and $\alpha_2 = 0.0650$

Hence, the eigenvalues are 0.0970 and 0.0650

Since, the Jacobian matrix's positive definite, then the steady state solution at $(T_1, T_2) = (0, 0)$ is unstable.

Evaluating the Jacobian matrix's elements J_{11}, J_{12}, J_{21} and J_{22} at

(2) $(T_1, T_2) = \left(\frac{\alpha_1}{\beta_1}, 0\right)$

$$T_1 = \frac{\alpha_1}{\beta_1}, T_2 = 0$$

$$J_{11} = \alpha_1 - 2\beta_1\left(\frac{\alpha_1}{\beta_1}\right) + r_1(0) = \alpha_1 - 2\alpha_1 + 0 = -\alpha_1$$

$$J_{12} = r_1(0) = 0$$

$$J_{21} = r_2\left(\frac{\alpha_1}{\beta_1}\right) = \frac{\alpha_1 r_2}{\beta_1}$$

$$J_{22} = \alpha_2 - 2\beta_2(0) + r_2\left(\frac{\alpha_1}{\beta_1}\right) = \alpha_2 - 0 + \frac{\alpha_1 r_2}{\beta_1} = \alpha_2 + \frac{\alpha_1 r_2}{\beta_1}$$

Let the Jacobian matrix at $(\ {}_1,\ {}_2) = \left(\frac{\alpha_1}{\beta_1}, 0\right)$ be J

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

$$J = \begin{pmatrix} -\alpha_1 & 0 \\ \frac{\alpha_1 r_2}{\beta_1} & \alpha_2 + \frac{\alpha_1 r_2}{\beta_1} \end{pmatrix}$$

We use the characteristics equation $|J - \lambda I| = 0$ to find the two eigenvalues.

Let $I = \lambda I$, where I is the identity matrix.

$$\therefore \quad \lambda I = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\therefore \quad J - \lambda I = \begin{pmatrix} -\alpha_1 & 0 \\ \frac{\alpha_1 r_2}{\beta_1} & \alpha_2 + \frac{\alpha_1 r_2}{\beta_1} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -(\alpha_1 + \lambda) & 0 \\ \frac{\alpha_1 r_2}{\beta_1} & (\alpha_2 + \frac{\alpha_1 r_2}{\beta_1} - \lambda) \end{pmatrix}$$

$$\therefore \quad |J - \lambda I| = \begin{vmatrix} -(\alpha_1 + \lambda) & 0 \\ \frac{\alpha_1 r_2}{\beta_1} & (\alpha_2 + \frac{\alpha_1 r_2}{\beta_1} - \lambda) \end{vmatrix}$$

$$= -(\alpha_1 + \lambda)(\alpha_2 + \frac{\alpha_1 r_2}{\beta_1} - \lambda) - \left(\frac{\alpha_1 r_2}{\beta_1}\right)(0)$$

$$\therefore \quad |J - \lambda I| = -(\alpha_1 + \lambda)\left(\alpha_2 + \frac{\alpha_1 r_2}{\beta_1} - \lambda\right) - 0$$

$$\therefore \quad |J - \lambda I| = -(\alpha_1 + \lambda)(\alpha_2 + \frac{\alpha_1 r_2}{\beta_1} - \lambda)$$

But $|J - \lambda I| = 0$

$$\therefore \quad -(\alpha_1 + \lambda)\left(\alpha_2 + \frac{\alpha_1 r_2}{\beta_1} - \lambda\right) = 0$$

$\therefore \quad (\alpha_1 + \lambda)\left(\alpha_2 + \frac{\alpha_1 r_2}{\beta_1} - \lambda\right) = 0$

It implies that $(\alpha_1 + \lambda) = 0$ or $\left(\alpha_2 + \frac{\alpha_1 r_2}{\beta_1} - \lambda\right) = 0$

Therefore $\lambda = -\alpha_1$ or $\lambda = \alpha_2 + \frac{\alpha_1 r_2}{\beta_1}$

Where $\alpha_1 = 0.097$, $r_2 = 0.0008$, $\beta_1 = 0.0036$ and $\alpha_2 = 0.0650$

Hence, the eigenvalues are α_1 and $2 \quad \frac{\alpha_1 r_2}{\beta_1}$

Since, the Jacobian matrix is negative indefinite, then the steady state solution at

$(T_1 , T_2) = \left(\frac{\alpha_1}{\beta_1} , 0\right)$ is unstable.

Evaluating the Jacobian matrix;s elements J_{11} , J_{12}, J_{21} and J_{22} at

(3) $(T_1 , T_2) = \left(0 , \frac{\alpha_2}{\beta_2}\right)$

$$T_1 = 0 , T_2 = \frac{\alpha_2}{\beta_2}$$

$$J_{11} = \alpha_1 - 2\beta_1(0) + r_1\left(\frac{\alpha_2}{\beta_2}\right) = \alpha_1 - 0 + \frac{\alpha_2 r_1}{\beta_2} = \alpha_1 + \frac{\alpha_2 r_1}{\beta_2}$$

$$J_{12} = r_1\left(\frac{\alpha_2}{\beta_2}\right) = \frac{\alpha_2 r_1}{\beta_2}$$

$$J_{21} = r_2(0) = 0$$

$$J_{22} = \alpha_2 - 2\beta_2\left(\frac{\alpha_2}{\beta_2}\right) + r_2(0) = \alpha_2 - 2\alpha_2 + 0 = -\alpha_2$$

Let the Jacobian matrix at $(T_1 , T_2) = \left(0 , \frac{\alpha_2}{\beta_2}\right)$ be J

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

$$J = \begin{pmatrix} \alpha_1 + \frac{\alpha_2 r_1}{\beta_2} & \frac{\alpha_2 r_1}{\beta_2} \\ 0 & -\alpha_2 \end{pmatrix}$$

We use the characteristics equation $|J - \lambda I| = 0$ to find the two eigenvalues.

Let $I = \lambda I$, where I is the identity matrix.

$$\therefore \quad \lambda I = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\therefore \quad J - \lambda I = \begin{pmatrix} \alpha_1 + \frac{\alpha_2 r_1}{\beta_2} & \frac{\alpha_2 r_1}{\beta_2} \\ 0 & -\alpha_2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} (\alpha_1 + \frac{\alpha_2 r_1}{\beta_2} - \lambda) & \frac{\alpha_2 r_1}{\beta_2} \\ 0 & -(\alpha_2 + \lambda) \end{pmatrix}$$

$$\therefore \quad |J - \lambda I| = \begin{vmatrix} (\alpha_1 + \frac{\alpha_2 r_1}{\beta_2} - \lambda) & \frac{\alpha_2 r_1}{\beta_2} \\ 0 & -(\alpha_2 + \lambda) \end{vmatrix}$$

$$= -(\alpha_1 + \frac{\alpha_2 r_1}{\beta_2} - \lambda)(\alpha_2 \quad \lambda \quad \left(\frac{\alpha_2 r_1}{\beta_2}\right) \; 0$$

$\therefore \quad |J - \lambda \mathrm{I}| = -(\alpha_1 + \frac{\alpha_2 r_1}{\beta_2} - \lambda)(\alpha_2 + \lambda) - 0$

$\therefore \quad |J - \lambda \mathrm{I}| = -(\alpha_1 + \frac{\alpha_2 r_1}{\beta_2} - \lambda)(\alpha_2 + \lambda)$

But $|J - \lambda \mathrm{I}| = 0$

$\therefore \quad -(\alpha_1 + \frac{\alpha_2 r_1}{\beta_2} - \lambda)(\alpha_2 + \lambda) = 0$

$\therefore \quad (\alpha_1 + \frac{\alpha_2 r_1}{\beta_2} - \lambda)(\alpha_2 + \lambda) = 0$

It implies that $(\alpha_1 + \frac{\alpha_2 r_1}{\beta_2} - \lambda) = 0$ or $(\alpha_2 + \lambda) = 0$

Therefore $\lambda = \alpha_1 + \frac{\alpha_2 r_1}{\beta_2}$ or $\lambda = -\alpha_2$

Where $\alpha_1 = 0.097$, $r_1 = 0.0012$, $\beta_1 = 0.0024$ and $\alpha_2 = 0.0650$

Hence, the eigenvalues are $\alpha_1 + \frac{\alpha_2 r_1}{\beta_2}$ and $-\alpha_2$

Since, the Jacobian matrix is negative indefinite, then the steady state solution at

$(T_1, T_2) = \left(0, \frac{\alpha_2 r_1}{\beta_2}\right)$ is unstable.

Evaluating the Jacobian matrix's elements J_{11} , J_{12}, J_{21} and J_{22} at (4), $(T_1, T_2) = \left(\frac{\alpha_1\beta_2 + r_1\alpha_2}{\beta_1\beta_2 - r_1 r_2}, \frac{\alpha_2\beta_1 + r_2\alpha_1}{\beta_1\beta_2 - r_1 r_2}\right) = (40.4688, 40.5729)$

$$T_1 = \frac{\alpha_1\beta_2 + r_1\alpha_2}{\beta_1\beta_2 - r_1 r_2}, T_2 = \frac{\alpha_2\beta_1 + r_2\alpha_1}{\beta_1\beta_2 - r_1 r_2}$$

$$\therefore \quad T_1 = \frac{\alpha_1\beta_2 + r_1\alpha_2}{\beta_1\beta_2 - r_1 r_2} = \frac{(0.097)(0.0024) + (0.0012)(0.065)}{(0.0036)(0.0024) - (0.0012)(0.0008)} = \frac{(2.328\times10^{-4}) + (0.7800\times10^{-4})}{(0.0864\times10^{-4}) - (0.0096\times10^{-4})}$$

$$\therefore \quad T_1 = \frac{(2.328 + 0.7800)\times10^{-4}}{(0.0864 - 0.0096)\times10^{-4}} = \frac{3.108}{0.0768} = 40.46875$$

$\therefore \quad T_1 = 40.46875$

$\therefore \quad T_1 \cong 40.4688$

Similarly,

$$T_2 = \frac{\alpha_2\beta_1 + r_2\alpha_1}{\beta_1\beta_2 - r_1 r_2}$$

$$\therefore \quad T_2 = \frac{\alpha_2\beta_1 + r_2\alpha_1}{\beta_1\beta_2 - r_1 r_2} = \frac{(0.065)(0.0036) + (0.0008)(0.097)}{(0.0036)(0.0024) - (0.0012)(0.0008)} = \frac{(2.34\times10^{-4}) + (0.776\times10^{-4})}{(0.0864\times10^{-4}) - (0.0096\times10^{-4})}$$

$$\therefore \quad T_2 = \frac{(2.3400 + 0.7760)\times10^{-4}}{(0.0864 - 0.0096)\times10^{-4}} = \frac{3.116}{0.0768} = 40.57291667$$

$\therefore \quad T_2 = 40.57291667$

$\therefore \quad T_2 \cong 40.5729$

We calculate the Jacobian matrix's elements as follows:

$$J_{11} = \alpha_1 - 2\beta_1 T_1 + r_1 T_2$$
$$= 0.0970 - 2(0.0036)(40.4688) + 0.0012(40.5729)$$
$$J_{11} = 0.0970 - 0.29137536 + 0.04868748$$
$$J_{11} = 0.14568748 - 0.29137536 = -0.14568788$$
$$J_{11} = -0.14568788$$
$$J_{11} \cong -0.1457$$
$$J_{12} = r_1 T_1 = 0.0012(40.4688) = 0.04856256$$
$$J_{12} = 0.04856256$$
$$J_{12} \cong 0.0486$$
$$J_{21} = r_2 T_2 = 0.0008(40.5729) = 0.03245832$$
$$J_{21} = 0.03245832$$
$$J_{21} \cong 0.0325$$
$$J_{22} = \alpha_2 - 2\beta_2 T_2 + r_2 T_1$$
$$= 0.0650 - 2(0.0024)(40.5729) + 0.0008(40.4688)$$
$$J_{22} = 0.0650 - 0.19474992 \quad 03237504$$
$$J_{22} = 0.09737504 - 0.19474992$$
$$J_{22} = -0.09737488$$
$$J_{22} \cong -0.0974$$

Let the Jacobian matrix at $(T_1, T_2) = (40.4688, 40.5729)$ be J

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$
$$J = \begin{pmatrix} -0.1457 & 0.0486 \\ 0.0325 & -0.0974 \end{pmatrix}$$

We use the characteristics equation $|J - \lambda \mathrm{I}| = 0$ to find the two eigenvalues.

Let $I = \lambda \mathrm{I}$, where I is the identity matrix.

$$\therefore \quad \lambda \mathrm{I} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$
$$\therefore J - \lambda \mathrm{I} = \begin{pmatrix} -0.1457 & 0.0486 \\ 0.0325 & -0.0974 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -(0.1457 + \lambda) & 0.0486 \\ 0.0325 & -(0.0974 + \lambda) \end{pmatrix}$$
$$\therefore \quad |J - \lambda \mathrm{I}| = \begin{vmatrix} -(0.1457 + \lambda) & 0.0486 \\ 0.0325 & -(0.0974 + \lambda) \end{vmatrix}$$
$$\therefore \quad |J - \lambda \mathrm{I}| = (0.1457 + \lambda)(0.0974 + \lambda) - (0.0486)(0.0325)$$
$$\therefore \quad |J - \lambda \mathrm{I}| = \lambda^2 + 0.1457\lambda + 0.0974\lambda + (0.1457)(0.0974) - 0.0015795$$
$$\therefore \quad |J - \lambda \mathrm{I}| = \lambda^2 + 0.2431\lambda + 0.01419118 - 0.0015795$$
$$\therefore \quad |J - \lambda \mathrm{I}| = \lambda^2 + 0.2431\lambda + 0.01261168$$

But $|J - \lambda \mathrm{I}| = 0$

Using quadratic equation formula,

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where, $a = 1, b = 0.2431, c = 0.01261168$

It implies that $\lambda = \frac{-0.2431 \pm \sqrt{0.2431^2 - 4(1)(0.01261168)}}{2(1)}$

$\therefore \quad \lambda = \frac{-0.2431 \pm \sqrt{0.05909761 - 0.05044672}}{2}$

$\therefore \quad \lambda = \frac{-0.2431 \pm \sqrt{0.00865089}}{2}$

$\therefore \quad \lambda = \frac{-0.2431 \pm 0.09301016074}{2}$

i.e. $\lambda = \frac{-0.2431 - 0.09301016074}{2}$ or $\lambda = \frac{-0.2431 - 0.09301016074}{2}$

$\therefore \quad \lambda = \frac{-0.3361101607}{2}$ or $\lambda = \frac{-0.1500898393}{2}$

$\therefore \lambda = -0.1680550804$ or $\lambda = -0.075004491963$

$\therefore \lambda \cong -0.1681$ or $\lambda \cong -0.0750$

Hence, the eigenvalues are -0.1681 and -0.0750

Since, the Jacobian matrix is negative definite, then the steady state solution at

$(T_1, T_2) = \left(\frac{\alpha_1\beta_2 + r_1\alpha_2}{\beta_1\beta_2 - r_1r_2}, \frac{\alpha_2\beta_1 + r_2\alpha_1}{\beta_1\beta_2 - r_1r_2} \right) = (40.4688, 40.5729)$ is stable.

RESULTS

On the application of our above mentioned method of analysis (ODE 45, numerical simulation) ,we here by represent the following results as displayed on TABLE 1 – 4.

T O S means type of stability. T_{1e} *and* T_{2e} are the coexistence steady state solutions for the first and second technology respectively.

$\beta_1 = 0.0036$, $\beta_2 = 0.0024$, $r_1 = 0.0012$, $r_1 = 0.0008$

$T_{1e} = \frac{\alpha_1\beta_2 + r_1\alpha_2}{\beta_1\beta_2 - r_1r_2}$, $T_{2e} = \frac{\alpha_2\beta_1 + r_2\alpha_1}{\beta_1\beta_2 - r_1r_2}$, $\alpha_1 = 0.0970$, $\alpha_2 = 0.0650$

We vary the intrinsic growth rates (α_1 and α_2) together to study its type of coexistence steady state solution and its type of stability.

Table 1 Quantifying the effect of decreasing the intra - competition coefficients (β_1 and β_2) together on the coexistence steady state solution and its type of stability, using ODE 45, numerical simulation.

Example	β_1	β_2	T_{1e}	T_{2e}	λ_1	λ_2	**T O S**
1	0.00360	0.00240	40.4688	40.5729	0.1680	0.0751	Stable
2	0.00018	0.00012	95.5243	95.1620	0.1078	0.0792	Degenerate
3	0.00036	0.00024	115.9341	115.6136	0.1484	0.0789	Degenerate
4	0.00054	0.00036	147.4922	147.2048	0.2113	0.0787	Degenerate
5	0.00072	0.00048	202.7344	202.4740	0.3216	0.0784	Degenerate
6	0.00090	0.00060	324.2857	324.0476	0.5645	0.0782	Degenerate
7	0.00108	0.00072	810.5263	810.3070	1.5367	0.0780	Degenerate

8	0.00126	0.00084	1620.7317	1620.9350	3.3260	0.0777	Stable
9	0.00144	0.00096	405.1136	405.3030	0.8950	0.0775	Stable
10	0.00162	0.00108	231.4590	231.6363	0.5479	0.0773	Stable
11	0.00180	0.00120	162.0000	162.1667	0.4091	0.0771	Stable
12	0.00198	0.00132	124.6009	124.7581	0.3346	0.0768	Stable
13	0.00216	0.00144	101.2277	101.3765	0.2880	0.0766	Stable
14	0.00234	0.00156	85.2364	85.3776	0.2562	0.0764	Stable
15	0.00252	0.00168	73.6070	73.7414	0.2332	0.0762	Stable

(Source: Authors' Matlab numerical simulated data)

Table 2 Quantifying the effect of decreasing the intra competition coefficients (together on the coexistence steady state solution and its type

Example	β_1	β_2	T_{1e}	T_{2e}	λ_1	λ_2	TOS
1	0.0027000	0.0018000	64.7692	64.8974	−0.2157	−0.0760	Stable
2	0.0028800	0.0019200	57.8256	57.9482	−0.2020	−0.0758	Stable
3	0.0030600	0.0020400	52.2263	52.3436	−0.1910	−0.0756	Stable
4	0.0032400	0.0021600	47.6153	47.7279	−0.1819	−0.0754	Stable
5	0.0034200	0.0022800	43.7522	43.8604	−0.1744	−0.0752	Stable
6	0.0034560	0.0023040	43.0536	43.1610	−0.1730	−0.0752	Stable
7	0.0034920	0.0023280	42.3769	42.4835	−0.1717	−0.0752	Stable
8	0.0035280	0.0023520	41.7212	41.8269	−0.1704	−0.0751	Stable
9	0.0035640	0.0023760	41.0854	41.1904	−0.1692	−0.0751	Stable
10	0.0035712	0.0023808	40.9606	41.0654	−0.1690	−0.0751	Stable
11	0.0035784	0.0023856	40.8365	40.9412	−0.1687	−0.0751	Stable
12	0.0035856	0.0023904	40.7132	40.8177	−0.1685	−0.0751	Stable
13	0.0035928	0.0023952	40.5906	40.6949	−0.1682	−0.0751	Stable

(Source: Authors' Matlab numerical simulated data)

Table 3 Quantifying the effect of increasing the intra competition coefficients (together on the coexistence steady state solution and its type of stability, using ODE 45, numerical simulation.

Example	β_1	β_2	T_{1e}	T_{2e}	λ_1	λ_2	TOS
1	0.003636	0.002424	39.8703	39.9737	0.1668	0.0750	Stable
2	0.003780	0.002520	37.6436	37.7440	0.1625	0.0749	Stable
3	0.003960	0.002640	35.1871	35.2840	0.1578	0.0747	Stable
4	0.004140	0.002760	33.0314	33.1250	0.1536	0.0745	Stable
5	0.004320	0.002880	31.1246	31.2152	0.1500	0.0744	Stable
6	0.004500	0.003000	29.4258	29.5136	0.1467	0.0742	Stable
7	0.004680	0.003120	27.9029	27.9879	0.1438	0.0741	Stable
8	0.004860	0.003240	26.5298	26.6123	0.1413	0.0739	Stable
9	0.005040	0.003360	25.2855	25.3656	0.1389	0.0738	Stable
10	0.005220	0.003480	24.1526	24.2305	0.1368	0.0736	Stable
11	0.005400	0.003600	23.1169	23.1926	0.1349	0.0735	Stable
12	0.005580	0.003720	22.1663	22.2401	0.1331	0.0733	Stable
13	0.005760	0.003840	21.2908	21.3627	0.1315	0.0732	Stable

(Source: Authors' Matlab numerical simulated data).

Table 4 Quantifying the effect of increasing the intra competition coefficients (together on the coexistence steady state solution and its type of stability, using ODE 45, numerical simulation.

Example	β_1	β_2	T_{1e}	T_{2e}	λ_1	λ_2	TOS
1	0.00594	0.00396	20.4819	20.5519	−0.1300	−0.0731	Stable
2	0.00612	0.00408	19.7321	19.8004	−0.1286	−0.0729	Stable
3	0.00630	0.00420	19.0353	19.1020	−0.1273	−0.0728	Stable
4	0.00648	0.00432	18.3860	18.4511	−0.1262	−0.0727	Stable
5	0.00666	0.00444	17.7795	17.8432	−0.1251	−0.0726	Stable
6	0.00684	0.00456	17.2118	17.2740	−0.1240	−0.0725	Stable
7	0.00702	0.00468	16.6792	16.7400	−0.1231	−0.0724	Stable
8	0.00720	0.00480	16.1786	16.2381	−0.1222	−0.0722	Stable
9	0.00738	0.00492	15.7071	15.7654	−0.1213	−0.0721	Stable
10	0.00756	0.00504	15.2623	15.3194	−0.1206	−0.0720	Stable
11	0.00774	0.00516	14.8421	14.8980	−0.1198	−0.0719	Stable
12	0.00792	0.00528	14.4443	14.4991	−0.1191	−0.0718	Stable
13	0.00810	0.00540	14.0673	14.1211	−0.1185	−0.0717	Stable

(Source: Authors' Matlab numerical simulated data)

DISCUSSION OF RESULTS

We vary the intra - competition coefficients (β_1 and β_2) together to study its type of coexistence steady state solution and its type of stability.
Looking at Table 1, when the intra - competition coefficients (β_1 and β_2) parameter values were varied together from 5% to 70%(example 2 to 15),we have found six (6) negative pairs unique coexistence steady state solutions which are dominantly degenerate having produced one positive

and one negative eignvalues that contributes to the decaying behavior of the solution trajectories over time. And also, we have found six (8) positive pairs unique coexistence steady state solutions which are dominantly stable having produced two negative eignvalues that contributes to the decaying behavior of the solution trajectories over time.

In the context of these analyses, the square Jacobian matrix is said to be indefinite negative. The intra – competition coefficients of the growth pattern increased monotonically from 0.00018 and 0.00012 to 0.00108 and 0.00072 respectively. And the volume of sales for the first technology (T_{1e}) and the second technology (T_{2e}) decreased monotonically from 95.5243 and 95.1620 to 810.5263 and 810.3070 respectively. And also, the intra – competition coefficients of the growth pattern increased monotonically from 0.000126 and 0.00084 to 0.00252 and 0.00168 respectively. And the volume of sales for the first technology (T_{1e}) and the second technology (T_{2e}) decreased monotonically from 1620.7317 and 1620.9350 to 73.6070 and 73.7414 respectively.

Looking at Table 2, when the intra - competition coefficients (β_1 and β_2) parameter values were varied together from 75% to 99.8%(example 1 to 13),we have found thirteen (13) positive pairs unique coexistence steady state solutions which are dominantly stable having produced two negative eignvalues that contributes to the decaying behavior of the solution trajectories over time.

In the context of these analyses, the square Jacobian matrix is said to be negative definite. The intra – competition coefficients the growth pattern increased monotonically from 0.00270 and 0.0018 to 0.0035928 and 0.0023952 respectively. And the volume of sales for the first technology (T_{1e}) and the second technology (T_{2e}) decreased monotonically from 64.7692 and 64.8974 to 40.5906 and 40.6949 respectively.

Looking at Table 3, when the intra - competition coefficients (β_1 and β_2) parameter values were varied together from 101% to 160%(example 1 to 13),we have found thirteen (13) positive pairs unique coexistence steady state solutions which are dominantly stable having produced two negative eignvalues that contributes to the decaying behavior of the solution trajectories over time.

In the context of these analyses, the square Jacobian matrix is said to be negative definite. The intra – competition coefficients the growth pattern increased monotonically from 0.003636 and 0.002424 to 0.00576 and 0.00384 respectively. And the volume of sales for the first technology (T_{1e}) and the second technology (T_{2e}) decreased monotonically from 39.8703 and 39.9737 to 21.2908 and 21.3627 respectively.

Looking at Table 4, when the intra - competition coefficients (β_1 and β_2) parameter values were varied together from 165% to 225%(example 1 to 13),we have found thirteen (13) positive pairs unique coexistence steady state solutions which are dominantly stable having produced two negative eignvalues that contributes to the decaying behavior of the solution trajectories over time.

In the context of these analyses, the square Jacobian matrix is said to be negative definite. The intra – competition coefficients the growth pattern increased monotonically from 0.00594 and 0.00396 to 0.00810 and 0.00540 respectively. And the volume of sales for the first technology (T_{1e}) and the second technology (T_{2e}) decreased monotonically from 20.4819 and 20.5519 to 14.0673 and 14.1211 respectively.

CONCLUSSION AND FURTHER RESEARCH

Without loss of generality, we have found six (6) negative pairs unique coexistence steady state solutions which are dominantly degenerate having produced one positive and one negative eignvalues, indicating the existence of fifty four (6) negative indefinite Jacobian matrices. And also, we have found fifty four (48) valid coexistence steady state solutions which are said to be dominantly stable in which the signs of two calculated eignvalues are negatives, indicating the existence of fifty four (48) negative definite Jacobian matrices. For these scenarios of results, the parameter value of β_1 ranges from the value of 0.00018 to 0.00810 whereas the parameter value of β_2 ranges from the value of 0.00012 to 0.00540. We have found few instances of degeneracy and bifurcation. But, we have not found any transition from a stable to an unstable steady state solution.

Our present application of a Matlab algorithm to construct a stability analysis of a mathematical model of two competing technologies can be extended to answer the following research questions that we did not explore in this present study:

1. Quantifying the type stability due to variation of the intrinsic growth rates between two competing technologies using Matlab algorithms.
2. Quantifying the type stability due to variation of the intra – competition coefficients between two competing technologies using Matlab algorithms.

REFERENCES

Ekaka-a, E. N., (2009). *Computational and Mathematical Modelling of Plant Species Interactions in Harsh Climate,* Doctorate's Thesis, Department of Mathematics, The University of Liverpool and The university of Chester, United Kingdom.

Ekaka-a, E. N., and Galadima, I. J., (2015). Numerical modeling of the impact of the carrying capacity (S) on the stability of two interacting mutualistic populations : issues for the recovery of stability loss*. Scientia Africana*, 14 (1).

Iyalla, I. B., (2018). *The Application of a Numerical Method to Predict Corrosion Penetration Rate Data Using a Newton – Gregory Forward Formula.* Master's Dissertation, Ignatius Ajuru University of Education, Rumuolumeni, Port Harcourt, Rivers State – Nigeria.

Mamat, M., Mada Sanjaya, W. S., Salleh, Z. and Ahmad, M. F., (2011). Numerical simulation dynamical model of three species food chain with Lotka-Volterra linear functional response. *Journal of Sustainability and Management.* 6(1), 44-50.

Nafo, N. M. and Ekaka-a, E. N., (2015). *Investigating the Effect of Fluctuating Economic Variable on the Stock Exchange Stability: Sustainable Development and Investment Planning.* Proceedings of the International Academic Conference for Sub – Sahara Transformation and Development. 3(2). University of Ilorin, 1000 Capacity Lecture Theatre Hall, Ilorin, Kwara State – Nigeria.

Zhang, G., McAdams, D. A., Shankar, V., and Mohammadi Darani, M. (March`23,2018). *"Technology Evolution Prediction Using Lotka – Volterra Equation"* ASME. J. Mec. Des. June 2018; 140(6): 061101. https://doi.org/10.1115/1.4039448