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A NOTE ON THE EXISTENCE AND UNIQUE OF SOLUTION OF FILTRATION COMBUSTION IN WET POROUS MEDIUM.

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Abstract
This paper establishes the criteria for the existence of unique solution of filtration combustion in a wet porous medium by analytical solution method.

Keywords: *Combustion, Eigenfunction expansion technique, Filtration Combustion, Porous medium.*

Introduction
Filtration combustion (FC) waves involve a heterogeneous exothermic reaction front propagating through a porous solid that reacts with a gas carrying oxidizer flowing through its pores (Aldushin, 2003). The propagation of combustion fronts in porous media is a subject of interest to a variety of applications, ranging from in situ combustion for the recovery of oil to catalyst regeneration, coal gasification, waste incineration, calcinations and agglomeration of ores, smoldering, and high-temperature synthesis of solid materials. The percolation of the oxidizing fluid plays a crucial role; therefore, such processes are often referred to generically as Filtration Combustion (FC). Several authors have studied the oxidation of crude oil with air injected in porous media; these include the work of Olayiwola *et al.* (2014) who presented a mathematical model for forward propagation of combustion front with Arrhenius kinetics through a porous medium with the reaction involving oxygen and solid fuel. They assume that the solid fuel depends on the space variable and that the amount of gas produced by the reaction is equal to the amount consumed by it and they proved the existence and uniqueness of solution of their model by actual solution method. The analytical solution of their model was provided via Homotopy perturbation method. They discovered that effect of Frank-kamenetskii number on the heat transfer and species consumption is of great

importance. Mailybaev *et al.* (2013) formulated a model for recovery of light oil by medium temperature oxidation. They considered two phase flow possessing a combustion front when a gaseous oxidizer (air) is injected into porous rock filled with light oil. The temperature of the medium is bounded by the boiling point of the liquid and, thus, relatively low. They disregarded the gas phase reactions. They observed that the initial period, the recovery curve is typical of gas displacement but after a critical amount of air has been injected the cumulative oil recovery increases linearly until all oil has been recovered, they conclude that oil recovery is independent of reaction rate parameters but recovery is much faster than for gas displacement and among their findings is that oil recovery is faster when the injected pressure is higher.
Bruining *et al.* (2009) developed a model of filtration combustion in wet porous medium. By considering a porous rock cylinder thermally insulated on the side filled with inert gas, liquid and solid fuel. An oxidizer was injected. They assumed that the amount of liquid is small, so its mobility is negligible, and that only a small part of the available space is occupied by solid fuel and liquid, so that changes of rock porosity in the reaction, evaporation, and condensation processes can be neglected. They neglected the dependence of thermal conductivity and diffusion coefficients on the temperature and gas compositions. They discovered that when the diffusion is dominant at the reaction layer, it lead the oxygen to extinction and also discovered two possible sequences of waves, and the internal structure of all waves was characterized. They compared the analytical results with direct numerical simulations. In this paper, the work of Bruining *et al.* (2009) is extended by incorporating temperature dependent thermal conductivity and diffusion coefficient. The criteria for the existence of unique solution of the model equations will be established. The properties of solution will be examined.

Model Formulation

Following Bruining *et al.*(2009), we consider a porous rock cylinder thermally insulated on the side and filled with vaporizable liquid, inert gas, and combustible solid fuel. An oxidizer (air) is injected. The liquid can be water or light oil, and the combustible solid can be coke. We assume that the amount of liquid is small, so its mobility is negligible. We assume that only a small part of the available space is occupied by solid fuel and liquid, so that we can neglect changes of rock porosity in the reaction, evaporation, and condensation processes. We assume that the solid, gas, and liquid are in local thermal equilibrium, so they have the same temperature. A one-dimensional model with time t and space coordinate x is considered. The heat transfer equation is

$$\rho c_g \frac{\partial}{\partial t}(T-T_{res})+\rho c_g u \frac{\partial}{\partial x}(T-T_{res})=\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right)+Q_r K_r Y n_f\, e^{-\frac{E_r}{RT}}-$$

$$Q_e k n_l\left(\frac{P_{atm}}{\rho RT} e^{-\frac{Q_e}{R}\left(\frac{1}{T}-\frac{1}{T_b}\right)}-X\right) \tag{1}$$

We consider a single component liquid (water), and denote by X its vapor molar fraction in the gas phase (mole of vapor/mole of gas). The gas has several components: oxygen, vapor, and passive (inert and combusted) gas. We denote the molar fractions of oxygen and passive gas in the gas-phase by Y and Z, respectively. Then, we write the mass balance equations for the components X, Y, Z as (see in Bruining *et al.(*2009).

$$\phi\rho\left(\frac{\partial X}{\partial t}+u\frac{\partial X}{\partial x}\right)=\phi\frac{\partial}{\partial x}\left(\rho D_X \frac{\partial X}{\partial x}\right)+k n_l\left(\frac{p_{atm}}{\rho RT} e^{-\frac{Q_e}{R}(\frac{1}{T}-\frac{1}{T_b})}-X\right) \tag{2}$$

$$\phi\rho\left(\frac{\partial Y}{\partial t}+u\frac{\partial Y}{\partial x}\right)=\phi\frac{\partial}{\partial x}\left(\rho D_Y \frac{\partial Y}{\partial x}\right)-\mu_o K_r Y n_f\, e^{-\frac{E_r}{RT}} \tag{3}$$

$$\phi\rho\left(\frac{\partial Z}{\partial t}+u\frac{\partial Z}{\partial x}\right)=\phi\frac{\partial}{\partial x}\left(\rho D_Z \frac{\partial Z}{\partial x}\right)+\mu_g K_r Y n_f\, e^{-\frac{E_r}{RT}} \tag{4}$$

As the solid fuel and the liquid do not move, their concentrations satisfy the equations for reaction and evaporation respectively:

$$\frac{\partial n_f}{\partial t}=\mu_f K_r Y n_f\, e^{-\frac{E_r}{RT}} \tag{5}$$

$$\frac{\partial n_l}{\partial t}=-k n_l\left(\frac{p_{atm}}{\rho RT} e^{-\frac{Q_e}{R}\left(\frac{1}{T}-\frac{1}{T_b}\right)}-X\right) \tag{6}$$

Where, ρ [mole/m^3] is the molar density of gas, T [k] is the temperature of reservoir when heated, T_{res} is the initial reservoir temperature, c_g is the heat capacity of rock, u [m/s] is the Darcy velocity of gas, λ [w/mk] is thermal conductivity of the porous medium, (Q_r and Q_e)[J/mole] are the heats enthalpies of combustion and evaporation of the solid and the liquid at reservoir temperature, K_r [1/s] is the pre exponential parameter, Y is the molar fraction of oxygen, X is the vapor molar fraction in the gas phase (mole of vapor/mole of gas), Z is the molar fraction of passive gas in the gas-phase, n_f Is the molar concentration of solid fuel, n_l Is the molar concentration of liquid, E_r [J/mole] is activation energy, R = 8.314[J/mole k] is the ideal gas constant, T_b is the boiling temperature of the liquid at atmospheric pressure p_{atm}, ϕ is the porosity, D_x [m²/s] is the diffusion coefficients for vapor of porous medium, D_y [m²/s] is the diffusion coefficients for oxygen of porous medium, D_z [m²/s] is the diffusion coefficients for passive gas in the gas-phase of porous medium, μ_f is the moles of solid fuel , μ_o is the moles of oxygen , μ_g is the moles of gaseous product.

Coordinate Transformation

The balance of mass can be eliminated by the means of streamline function (Olayiwola, 2015) giving by equation (7)

$$\eta(x.t) = \left(\rho^2\right)^{-\frac{1}{2}} \int_0^x \rho(x,t)ds \tag{7}$$

The coordinate transformation becomes,

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \eta} \tag{8}$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial t} = -u\frac{\partial}{\partial \eta} + \frac{\partial}{\partial t} \tag{9}$$

We make the additional assumptions that ρc_g , ρD ,and λ are constant. Although these assumptions could be relaxed in the future, they considerably simplify the equations. The equations (1) to (6) can be simplified as:

$$\rho c_g \frac{\partial}{\partial t}\left(T - T_{res}\right) = \frac{\partial}{\partial \eta}\left(\lambda \frac{\partial T}{\partial \eta}\right) + Q_r K_r Y n_f\, e^{-\frac{E_r}{RT}} - Q_e k n_l \left(\frac{p_{atm}}{\rho RT} e^{-\frac{Q_e}{R}\left(\frac{1}{T}-\frac{1}{Tb}\right)} - X\right)$$

(10)

$$\phi\rho \frac{\partial X}{\partial t} = \phi \frac{\partial}{\partial \eta}\left(\rho D_X \frac{\partial X}{\partial \eta}\right) + k n_l \left(\frac{p_{atm}}{\rho RT} e^{-\frac{Q_e}{R}\left(\frac{1}{T}-\frac{1}{Tb}\right)} - X\right)$$

(11)

$$\phi\rho \frac{\partial Y}{\partial t} = \phi \frac{\partial}{\partial \eta}\left(\rho D_Y \frac{\partial Y}{\partial \eta}\right) - \mu_o K_r Y n_f\, e^{-\frac{E_r}{RT}} \qquad (12)$$

$$\phi\rho \frac{\partial Z}{\partial t} = \phi \frac{\partial}{\partial \eta}\left(\rho D_Z \frac{\partial Z}{\partial \eta}\right) + \mu_g K_r Y n_f\, e^{-\frac{E_r}{RT}} \qquad (13)$$

The initial and boundary conditions were formulated as follows:
Initial condition:

At $t = 0$ and $\forall \eta$

$$\left.\begin{array}{l} T = \dfrac{RT_0^{\,2}}{E}\left(1 - \dfrac{\eta}{L}\right) + T_0, \\ X = X_0\left(1 - \dfrac{\eta}{L}\right), \qquad Y = Y_0\left(1 - \dfrac{\eta}{L}\right), \\ Z = Z_0\left(1 - \dfrac{\eta}{L}\right), \qquad n_f = n_{fres,} \quad n_l = n_{lres} \end{array}\right\} \qquad (14)$$

Boundary Condition:

$$\left.\begin{array}{ll} T\big|_{\eta=0} = T_1, & T\big|_{\eta=l} = T_0 \\ Y\big|_{\eta=0} = Y_{inj}, & Y\big|_{\eta=l} = 0 \\ X\big|_{\eta=0} = 0, & X\big|_{\eta=l} = 0 \\ Z\big|_{\eta=0} = 0, & Z\big|_{\eta=0} = 0 \end{array}\right\} \qquad (15)$$

Method of Solution

Here, we examine the properties of solution of the equations (10) – (15).

Existence and Uniqueness of Solution

Theorem 3.1: let $D_x = D_y = D_z = \frac{\lambda}{\rho c_g} = T_{res}$ = constant. Then there exists a unique solution of (10)-(13) satisfy (14) and (15).

Proof: Let $D_x = D_y = D_z = \frac{\lambda}{\rho c_g} = T_{res}$ = constant and

$$\psi = \left(\frac{\phi Q_e \mu_0}{c_g} X + \mu_0 T + \frac{\phi(Q_r + \mu_g)Y}{c_g} + \frac{\phi \mu_0}{c_g} Z \right)$$

Then, (10) to (13) becomes

$$\frac{\partial \psi}{\partial t} = D \frac{\partial^2 \psi}{\partial \eta^2} \tag{16}$$

$$\psi(\eta,0) = A\left(1 - \frac{\eta}{L}\right) + B, \quad \psi(0,t) = A_1, \quad \psi(L,t) = B \tag{17}$$

Using eigenfunction expansion technique, we obtain the solution of problem (16) and (17) as

$$\psi(\eta,t) = A_1 + \frac{\eta}{L}(B - A_1) + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left((-1)^n - (A+B)\left((-1)^n - 1\right)\right) e^{-D\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi}{L}\eta \tag{18}$$

Then, we have

$$T(\eta,t) = \frac{1}{\mu_0}\left(A_1 + \frac{\eta}{L}(B - A_1) + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left((-1)^n - (A+B)\left((-1)^n - 1\right)\right) e^{-D\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi}{L}\eta - \left(\frac{\phi Q_e \mu_0}{c_g} X + \frac{\phi(Q_r + \mu_g)}{c_g} Y + \frac{\phi \mu_0}{c_g} Z \right)\right) \tag{19}$$

$$X(\eta,t) = \frac{c_g}{\phi Q_e \mu_0}\left(A_1 + \frac{\eta}{L}(B - A_1) + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left((-1)^n - (A+B)\left((-1)^n - 1\right)\right) e^{-D\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi}{L}\eta - \left(\mu_0 T + \frac{\phi(Q_r + \mu_g)}{c_g} Y + \frac{\phi \mu_0}{c_g} Z \right)\right) \tag{20}$$

$$Y(\eta,t) = \frac{c_g}{\phi(Q_r + \mu_g)}\left(A_1 + \frac{\eta}{L}(B - A_1) + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left((-1)^n - (A+B)\left((-1)^n - 1\right)\right) e^{-D\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi}{L}\eta \right.$$

$$-\left(\mu_0 T+\frac{\phi Q_e\mu_0}{c_g}Y+\frac{\phi\mu_0}{c_g}Z\right)\Bigg)\Bigg) \tag{21}$$

$$Z(\eta,t)=\frac{c_g}{\phi\mu_0}\left(A_1+\frac{\eta}{L}(B-A_1)+\sum_{n=1}^{\infty}\frac{2}{n\pi}\left((-1)^n-(A+B)\left((-1)^n-1\right)\right)e^{-D\left(\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}\eta\right.$$

$$\left.-\left(\mu_0 T+\frac{\phi Q_e\mu_0}{c_g}X+\frac{\phi(Q_r+\mu_0)}{c_g}Y\right)\right) \tag{22}$$

Where, A= $\left(\frac{\phi Q_e\mu_0}{c_g}X_0+\frac{\mu_0 RT_0^2}{}+\frac{\phi(Q_r+\mu_g)Y_0}{c_g}+\frac{\phi\mu_0}{c_g}Z_0\right)$, $B=\mu_0 T_0$ $A_1=\left(\mu_0 T_1+\frac{\phi(Q_r+\mu_g)}{c_g}Y_{inj}\right)$

Hence, there exist a unique solution of problem (12)- (15). This completes the proof.

We shall now consider an alternative method for the existence of unique solution of the problem. Here, the dependence of thermal conductivity and diffusion coefficient on the temperature is taken into account by the mathematical expression:

$$\lambda=\lambda_0\left(\frac{T}{T_0}\right) \tag{23}$$

$$D=D_0\left(\frac{T}{T_0}\right) \tag{24}$$

Where λ_0 is the initial thermal conductivity, D_0 is the initial diffusion coefficient, and T_0 is the initial temperature of the medium.

Substituting (20), (21), (22) and (23) into (5) and (6), (10) – (13), we have

$$\rho c_g\frac{\partial}{\partial t}(T-T_{res})=\lambda_0\frac{\partial}{\partial\eta}\left(\frac{T}{T_0}\frac{\partial T}{\partial\eta}\right)+Q_rK_rn_f\left(\frac{c_g}{\phi(Q_r+\mu_g)}\left(A_1+\frac{\eta}{L}(B-A_1)+\right.\right.$$

$$\left.\left.\sum_{n=1}^{\infty}\frac{2}{n\pi}\left((-1)^n-(A+B)\left((-1)^n-1\right)\right)e^{-D\left(\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}\eta-\left(\mu_0 T+\frac{\phi Q_e\mu_0}{c_g}X+\frac{\phi\mu_0}{c_g}Z\right)\right)\right)e^{-\frac{E_r}{RT}}-$$

$$kn_l\left(\frac{p_{atm}}{\rho RT}e^{-\frac{Q_e}{R}\left(\frac{1}{T}-\frac{1}{Tb}\right)}-X\right) \tag{25}$$

$$\phi\rho\frac{\partial X}{\partial t}=\phi\rho D_0\frac{\partial}{\partial\eta}\left(\frac{T}{T_0}\frac{\partial X}{\partial\eta}\right)+kn_l\left(\frac{p_{atm}}{\rho RT}e^{-\frac{Q_e}{R}\left(\frac{1}{T}-\frac{1}{Tb}\right)}-X\right)$$

(26)

$$\phi\rho\frac{\partial Y}{\partial t}=\phi\rho D_0\frac{\partial}{\partial\eta}\left(\frac{T}{T_0}\frac{\partial Y}{\partial\eta}\right)-\mu_0K_rn_f\left(\frac{c_g}{\phi(Q_r+\mu_g)}\left(A_1+\frac{\eta}{L}(B-A_1)+\right.\right.$$

$$\left.\left.\sum_{n=1}^{\infty}\frac{2}{n\pi}\left((-1)^n-(A+B)((-1)^n-1)\right)e^{-D\left(\frac{n\pi}{L}\right)^2t}\sin\frac{n\pi}{L}\eta-\left(\mu_0T+\frac{\phi Q_e\mu_0}{c_g}X+\frac{\phi\mu_0}{c_g}Z\right)\right)\right)e^{-\frac{E_r}{RT}}$$

(27)

$$\phi\rho\frac{\partial Z}{\partial t}=\phi\rho D_0\frac{\partial}{\partial\eta}\left(\frac{T}{T_0}\frac{\partial Z}{\partial\eta}\right)+\mu_gK_rn_f\left(\frac{c_g}{\phi(Q_r+\mu_g)}\left(A_1+\frac{\eta}{L}(B-A_1)+\right.\right.$$

$$\left.\left.\sum_{n=1}^{\infty}\frac{2}{n\pi}\left((-1)^n-(A+B)((-1)^n-1)\right)e^{-D\left(\frac{n\pi}{L}\right)^2t}\sin\frac{n\pi}{L}\eta-\left(\mu_0T+\frac{\phi Q_e\mu_0}{c_g}X+\frac{\phi\mu_0}{c_g}Z\right)\right)\right)e^{-\frac{E_r}{RT}}$$

(28)

$$\phi\rho\frac{\partial n_f}{\partial t}=\mu_fK_rn_f\left(\frac{c_g}{\phi(Q_r+\mu_g)}\left(A_1+\frac{\eta}{L}(B-A_1)+\sum_{n=1}^{\infty}\frac{2}{n\pi}\left((-1)^n-(A+B)((-1)^n-1)\right)e^{-D\left(\frac{n\pi}{L}\right)^2t}\sin\frac{n\pi}{L}\eta\right.\right.$$

$$\left.\left.-\left(\mu_0T+\frac{\phi Q_e\mu_0}{c_g}X+\frac{\phi\mu_0}{c_g}Z\right)\right)\right)e^{-\frac{E_r}{RT}} \qquad (29)$$

Non – dimensionalization

Here we non-dimensionalized equation (25) to (29) using the following dimensionless variables

$$\left.\begin{array}{l} X^1=\dfrac{X}{X_0},\quad Y^1=\dfrac{Y}{Y_0},\quad Z^1=\dfrac{Z}{Z_0},\quad \theta=\dfrac{E}{RT_0}(T-T_0),\\ t^1=\dfrac{t}{t_0},\quad \eta^1=\dfrac{\eta}{L},\quad n_f^{\,1}=\dfrac{n_f}{n_{fres}},\quad n_l^{\,1}=\dfrac{n_l}{n_{lres}},\quad \varepsilon=\dfrac{RT_0}{E}\end{array}\right\}$$

(30)

And we obtain

$$\frac{\partial\theta}{\partial t}=\lambda_1\frac{\partial}{\partial\eta}\left((1+\varepsilon\theta)\frac{\partial\theta}{\partial\eta}\right)+\delta\left(a_1(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1e^{-P_{em}n^2\pi^2t^1}\sin n\pi\eta-(b_1(1+\varepsilon\theta)+a_2X+b_2Z))\right)n_f\,e^{\frac{\theta}{1+\varepsilon\theta}}-$$

$$\alpha\left(a_3\frac{e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)}}{1+\varepsilon\theta}-X\right) \qquad (31)$$

$$\frac{\partial X}{\partial t} = D_1 \frac{\partial}{\partial \eta}\left((1+\varepsilon\theta)\frac{\partial X}{\partial \eta}\right) + \alpha_2 n_l \left(\alpha_3 \frac{e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)}}{1+\varepsilon\theta} - X\right)$$

(32)

$$\frac{\partial Y}{\partial t} = D_1 \frac{\partial}{\partial \eta^1}\left((1+\varepsilon\theta)\frac{\partial Y}{\partial \eta}\right) - \gamma\Big(a_1\Big(A_1 + (B-A_1)\eta^1 + \sum_{n=1}^{\infty} B_1 e^{-P_{em}n^2\pi^2 t^1} \sin n\pi\eta -$$

$$\big(b_1(1+\varepsilon\theta) + a_2 X + b_2 Z\big)\Big)\Big) n_f \, e^{\frac{\theta}{1+\varepsilon\theta}} \quad (33)$$

$$\frac{\partial Z}{\partial t} = D_1 \frac{\partial}{\partial \eta^1}\left((1+\varepsilon\theta)\frac{\partial Z}{\partial \eta}\right) + \gamma_2\Big(a_1\Big(A_1 + (B-A_1)\eta^1 + \sum_{n=1}^{\infty} B_1 e^{-P_{em}n^2\pi^2 t^1} \sin n\pi\eta - \big(b_1(1+\varepsilon\theta) + a_2 X + b_2 Z\big)\Big)\Big) n_f \, e^{\frac{\theta}{1+\varepsilon\theta}}$$

(34)

$$\frac{\partial n_f}{\partial t} = \gamma_3 n_f^1 \Big(a_1\Big(A_1 + (B-A_1)\eta^1 + \sum_{n=1}^{\infty} B_1 e^{-P_{em}n^2\pi^2 t} \sin n\pi\eta - \big(b_1(1+\varepsilon\theta) + a_2 X + b_2 Z\big)\Big)\Big) e^{\frac{\theta}{1+\varepsilon\theta}}$$

(35)

$$\frac{\partial n_l}{\partial t} = -t_0 k X_0 n_l^1 \left(\frac{P_{atm}}{\rho R T_0 X_0} \frac{e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)}}{1+\varepsilon\theta} - X\right) \quad (36)$$

Substituting (30) into the initial and boundary condition (5) and (6), we obtain

$$\left.\begin{array}{lll} \theta(\eta,0) = 1-\eta, & \theta(0,t) = b_3, & \theta(1,t) = 0 \\ X(\eta,0) = 1-\eta, & X(0,t) = 0, & X(1,t) = 0 \\ Y(\eta,0) = 1-\eta, & Y(0,t) = 1, & Y(1,t) = 0 \\ Z(\eta,0) = 1-\eta, & Z(0,t) = 0 & Z(1,t) = 0 \\ n_f = 1, & n_l = 1 & \end{array}\right\} \quad (37)$$

Where, $B_1 = \frac{2}{n\pi}\left((-1)^n - (A+B)\left((-1)^n - 1\right)\right)$, $a = \frac{Q_e \varepsilon}{RT_0}$, $b = \frac{T_b - T_0}{\varepsilon T_0}$, $\lambda_1 = \frac{\lambda_0 t_0}{\rho c_g L^2}$, $\delta = \frac{t_0 Q_r k_r n_{fres}}{\rho c_g \varepsilon T_0} e^{-\frac{Er}{RT_0}}$, $P_{em} = \frac{Dt_0}{L^2}$, $a_1 = \frac{c_g}{\phi(Q_r + \mu_0)}$, $b_1 = \mu_0 T_0$, $a_2 = \frac{\phi Q_e \mu_0 X_0}{c_g}$, $b_2 = \frac{\phi \mu_0 Z_0}{c_g}$, $\alpha = \frac{t_0 Q_e k n_{lres} X_0}{\rho c_g \varepsilon T_0}$, $a_3 = \frac{P_{atm}}{\rho R T_0 X_0}$, $D_1 = \frac{t_0 D_0}{L^2}$, $\alpha_2 = \frac{t_0 k n_{lres}}{\phi \rho}$,

$$\gamma = \frac{t_0 \mu_0 k_r n_{fres} n}{\phi \rho Y_0} e^{-\frac{Er}{RT_0}} \; \gamma_2 = \frac{t_0 \mu_g k_r n_{fres}}{\phi \rho Y_0} e^{-\frac{Er}{RT_0}}, \qquad \gamma_3 = t_0 \mu_f k_r e^{-\frac{Er}{RT_0}},$$

$$\beta = t_0 k X_0, \quad b_3 = \varepsilon \frac{T_1 - T_0}{\varepsilon T_0}.$$

Properties of Solution

To examine the properties of solution, we consider the following asymptotic expansion of temperature θ and concentrations $X, Y, Z, n_f \;\; and \; n_l \;\; in\, \varepsilon$.
Let

$$\left.\begin{aligned} \theta &= \theta_0 + \varepsilon\theta_1 + \ldots \\ X &= X_0 + \varepsilon X_1 + \ldots \\ Y &= Y_0 + \varepsilon Y_1 + \ldots \\ Z &= Z_0 + \varepsilon Z_1 + \ldots \\ n_f &= n_{f0} + \varepsilon n_{f1} + \ldots \\ nl &= n_{l0} + \varepsilon n_{l1} + .. \end{aligned}\right\}. \tag{38}$$

Collectinglike power of ε, we have for

ε^0:

$$\frac{\partial \theta_0}{\partial t} = \lambda_1 \frac{\partial^2 \theta}{\partial \eta^2} + \delta\left(a_1\left(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1 e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - (b_1 + a_2 X_0 + b_2 Z_0)\right)\right) n_{f0} e^{\theta_0} -$$

$$\alpha n_{l0}\left(-aba_3 e^{a\theta_0} - X_0\right) \tag{39}$$

$$\frac{\partial X_0}{\partial t} = D_1 \frac{\partial^2 X_0}{\partial \eta^2} + \alpha_2 n_{l0}\left(-aba_3 e^{a\theta_0} - X_0\right) \tag{40}$$

$$\frac{\partial Y_0}{\partial t} = D_1 \frac{\partial^2 Y_0}{\partial \eta^2} - \gamma\left(a_1\left(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1 e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - (b_1 + a_2 X_0 + b_2 Z_0)\right)\right) n_{f0} e^{\theta_0} \tag{41}$$

$$\frac{\partial Z_0}{\partial t} = D_1 \frac{\partial^2 Z_0}{\partial \eta^2} + \gamma_2\left(a_1\left(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1 e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - (b_1 + a_2 X_0 + b_2 Z_0)\right)\right) n_{f0} e^{\theta_0} \tag{42}$$

This question of existence and uniqueness of solutions to these equations has been addressed by Ayeni (1978) who consider a similar set of equations and showed among other results that existence and uniqueness are somewhat well known. In his work, he studied the following system of parabolic equations.

$$\left.\begin{aligned}\frac{\partial \phi}{\partial t} &= \Delta\phi + f(x,t,\phi,u,v) \qquad x \in R^n,\ t>0 \\ \frac{\partial u}{\partial t} &= \Delta u + g(x,t,\phi,u,v) \qquad x \in R^n,\ t>0 \\ \frac{\partial v}{\partial t} &= \Delta v + f(x,t,\phi,u,v) \qquad x \in R^n,\ t>0\end{aligned}\right\} \tag{43}$$

$$\phi(x,0) = f_0(x)$$

$$u(x,0) = g_0(x)$$

$$v(x,0) = h_0(x)$$

$$x = (x_1, x_2, x_3, \ldots, x_n)$$

(S.1): $f_0(x), g(x)$ and $h_0(x)$ are bounded for $x \in R^n$. Each has at most countable number of discontinuities.

(S.2): f, g, h satisfies the uniform Lipschitz condition

$$|\varphi(x,t,\phi_1,u_1,v_1) - \varphi(x,t,\phi_2,u_2,v_2)| \le M(|\phi_1-\phi_2| + |u_1-u_2| + |v_1-v_2|),\ (x,t)\in G$$

Where;

$$G = \{(x,t) : x \in R^n, 0 < t < \tau\}.$$

Our proof of existence of unique solution of the system of parabolic equations (39) to (42) will be analogous to his proof.

Theorem 3.2: There exists a unique solution $\theta_0(\eta,t), X_0(\eta,t), Y_0(\eta,t), and\ Z_0(\eta,t)$ of equation (39), (40), (41) and (42) which satisfy (37).

Lemma 3.1 (Ayeni (1978))

Let $(f_{0,}\ g_{0,}\ h_{0,}\ j_0)$ and $(f_,\ g_,\ h_,\ j)$ satisfy $(S.1)\ and\ (S.2)$ respectively. Then there exists a solution of problem (39), (40), (41) and (42).

Proof: see Ayeni (1978)

Proof of theorem 3.2

We rewrite equations (40), (41), (42) and (43) as

$$\frac{\partial \theta_0}{\partial t} = \lambda_1 \frac{\partial^2 \theta_0}{\partial \eta^2} + f(\eta, t, \theta_0, X_0, Y_0, Z_0) \qquad \eta \in R^n, \ t > 0$$

(44)

$$\frac{\partial X_0}{\partial t} = D_1 \frac{\partial^2 X_0}{\partial \eta^2} + g(\eta, t, \theta_0, X_0, Y_0, Z_0) \qquad \eta \in R^n, \ t > 0$$

(45)

$$\frac{\partial Y_0}{\partial t} = D_1 \frac{\partial^2 Y_0}{\partial \eta^2} + h(\eta, t, \theta_0, X_0, Y_0, Z_0) \qquad \eta \in R^n, \ t > 0$$

(46)

$$\frac{\partial Z_0}{\partial t} = D_1 \frac{\partial^2 Z_0}{\partial \eta^2} + j(\eta, t, \theta_0, X_0, Y_0, Z_0) \qquad \eta \in R^n, \ t > 0$$

(47)

Where

$$f(\eta, t, \theta_0, X_0, Y_0, Z_0) = \delta\left(a_1\left(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1 e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - (b_1 + a_2 X_0 + b_2 Z_0)\right)\right) n_{f0} e^{\theta_0} -$$

$$\alpha n_{l0}\left(-aba_3 e^{a\theta_0} - X_0\right).$$

$$g(\eta, t, \theta_0, X_0, Y_0, Z_0) = \alpha_2 n_{l0}\left(-aba_3 e^{a\theta_0} - X_0\right).$$

$$h(\eta, t, \theta_0, X_0, Y_0, Z_0) = -\gamma\left(a_1\left(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1 e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - (b_1 + a_2 X_0 + b_2 Z_0)\right)\right) n_{f0} e^{\theta_0}$$

$$j(\eta, t, \theta_0, X_0, Y_0, Z_0) = \gamma_2\left(a_1\left(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1 e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - (b_1 + a_2 X_0 + b_2 Z_0)\right)\right) n_{f0} e^{\theta_0}$$

Ignoring the second term at the right hand side, the fundamental solution of equations (44), (45), (46) and (47) are (see Toki and Tokis (2007)).

$$F(\eta, t) = \frac{\eta}{2\lambda_1 \pi^{1/2} t^{3/2}} e^{-\frac{\eta^2}{4\lambda_1 t}}$$

$$G(\eta, t) = \frac{\eta}{2D_1 \pi^{1/2} t^{3/2}} e^{-\frac{\eta^2}{4D_1 t}}$$

$$H(\eta, t) = \frac{\eta}{2D_1 \pi^{1/2} t^{3/2}} e^{-\frac{\eta^2}{4D_1 t}}$$

$$J(\eta,t)=\frac{\eta}{2D_1\pi^{1/2}t^{3/2}}e^{-\frac{\eta^2}{4D_1t}}$$

Clearly,

$$f(\eta,\, t,\, \theta_0,\, X_0,\, Y_0,\, Z_0)=\delta\left(a_1\left(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1\,e^{-P_{em}n^2\pi^2t}\sin n\pi\eta-(b_1+a_2X_0+b_2Z_0)\right)\right)n_{f0}\,e^{\theta_0}-$$

$$\alpha n_{l0}\left(-aba_3\,e^{a\theta_0}-X_0\right),\quad g(\eta,\, t,\, \theta_0,\, X_0,\, Y_0,\, Z_0)=\alpha_2 n_{l0}\left(-aba_3\,e^{a\theta_0}-X_0\right),$$

$$h(\eta,\, t,\, \theta_0,\, X_0,\, Y_0,\, Z_0)=-\gamma\left(a_1\left(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1\,e^{-P_{em}n^2\pi^2t}\sin n\pi\eta-(b_1+a_2X_0+b_2Z_0)\right)\right)n_{f0}\,e^{\theta_0}$$

and

$$j(\eta,\, t,\, \theta_0,\, X_0,\, Y_0,\, Z_0)=\gamma_2\left(a_1\left(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1\,e^{-P_{em}n^2\pi^2t}\sin n\pi\eta-(b_1+a_2X_0+b_2Z_0)\right)\right)n_{f0}\,e^{\theta_0}$$

are lipschitz continues. Hence by theorem 3.3, the result follows. This completes the proof.

Conclusion

In this study, we have established the criteria for the existence of unique solution and examined the properties of solution of equations governing the filtration combustion in a wet porous medium by actual solution method. Our analysis showed that the solutions exist and it is unique. Future researchers may investigate the influence of dimensionless parameter such as scaled thermal conductivity λ_1, species diffusion coefficient D_1, Frank kamenetskii parameter and δ peclet mass number p_{em} on the filtration combustion.

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