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STABILITY ANALYSIS OF THE DISEASE FREE EQUILIBRIUM STATE AND SENSITIVITY ANALYSIS OF PARAMETERS WITH RESPECT TO THE BASIC REPRODUCTIVE NUMBER OF A DETERMINISTIC MODEL OF LASSA FEVER

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ABSTRACT
In this study, we carry out the stability analysis of the disease free equilibrium state and sensitivity analysis of parameters with respect to the basic reproductive number of the model presented in (Enagi and Muhammed, 2019).The disease-free equilibrium state is stable if $\frac{\lambda \wedge_2}{(\nu+\mu_2)} < (\nu+\mu_2)$ *and* $R_0 \epsilon (R_{0H}, R_{0R}) < 1$ *otherwise unstable. We also analyzed the global stability of the disease-free-equilibrium using Castillo-Chavez, Feng and Huang approach. The result shows that the disease-free-equilibrium state is also globally asymptotically stable. The most sensitive parameters to the humans' basic reproduction number are the natural birth rate,* Λ_1 *and contact rate* β*, while the most sensitive parameters to the rodents' basic reproduction number are the natural birth rate,* Λ_2 *and contact rate,* λ.

Keywords: *Lassa Fever, Equilibrium State, Stability Analysis, Sensitivity Analysis, Basic Reproduction Number.*

INTRODUCTION
Lassa fever is a zoonotic disease that can be transmitted to humans from infected rodents known asmastomyms rats. Person-to-Person transmission of Lassa virus also occurs through direct contact with the blood, urine, faeces, or other bodily secretions or fluids of an infected person (World health Organization, 2017).Lassa fever is mostly common in West African countries such as Nigeria, Ghana, Guinea, Liberia, Mali, Sierra Leone, and, Benin (World Health organization, 2017).There is currently no US approved vaccine for Lassa fever

but it can be treated using Ribavirin which is effective if administered during the early stage of infectiousness (Omale *et al*, 2014).

Lassa fever is a viral hemorrhagic fever whose symptoms include fever, sore throat, headache, facial swelling, muscle fatigue, vomiting, muscles pain, cough, bleeding, meningitis, and hypertension(Omilabu *et al*, 2005).In some cases neurological problems such hearing loss may be transient or permanent and tremors have been described (Omilabu *et al*, 2005).

Over the years Mathematical models of Lassa virus has been developed and analysed. These includes Okuonghae *et al.*, (2006) who developed and analyzed an SIS model for the transmission of Lassa virus. They obtained the equilibrium states of their model and analyzed them for stability. They gave the conditions for the disease to be endemic and also calculated the reproductive number for their model. They concluded that the best strategies to stop the spread of the diseases are isolation policy and the control of rodents carrying the virus. But they didn't consider treatment and recovered classes.

Bawa *et al.,(2014)* formulaed a mathematical model which incorporated vital dynamics, standard incidence, disease induced death due to human infection, and aerosol (airborne) transmissions. Their analysis revealed that the disease can be control if the basic reproduction number R_0 is strictly less than unity. Their work didn't take into account treated and recovered humans

Tolulope *et al., (2015)* formulated a deterministic model for Lassa fever transmission by in-corporating quarantine and permanent immunity. The model was validated for existence and uniqueness of solutions. the basic reproduction number was computed and was used with Lyapunov function to analyze the disease-free equilibrium state for stability.

In this study, we carry out the stability analysis of the disease free equilibrium state and sensitivity analysis of parameters with respect to the basic reproductive number of the model presented in (Enagi and Muhammed 2019).

METHODOLOGY

MODEL EQUATIONS

Recall that the model equations presented by (Enagi and Muhammed, 2019) are as follows:

$$\frac{dS_H}{dt} = \wedge_1 - \beta(I_R + I_H + A_H + \delta T_H)S_H - \mu_1 S_H \quad (1)$$

$$\frac{dE_H}{dt} = \beta(I_R + I_H + A_H + \delta T_H)S_H - (\alpha + \mu_1)E_H \quad (2)$$

$$\frac{dA_H}{dt} = (1-\rho)\alpha E_H - (\eta + \phi + \mu_1)A_H \quad (3)$$

$$\frac{dI_H}{dt} = \rho\alpha E_H - (\gamma + \phi + \mu_1)I_H \quad (4)$$

$$\frac{dT_H}{dt} = \gamma I_H + \eta A_H - (\kappa + \phi + \mu_1)T_H \quad (5)$$

$$\frac{dR_H}{dt} = \kappa T_H - \mu_1 R_H \quad (6)$$

$$\frac{dS_R}{dt} = \wedge_2 - \lambda S_R I_R - (\nu + \mu_2)S_R \quad (7)$$

$$\frac{dI_R}{dt} = \lambda S_R I_R - (\nu + \mu_2)I_R \quad (8)$$

With Variables and Parameters described below

Variable /Parameter	Description
A_H	Number of asymptomatic infected humans
E_H	Number of exposed humans
I_H	Number of symptomatic infected humans
I_R	Number of infected humans
N_H	Total number of the humans' population
N_H	Total number of the reservoir' population
R_H	Number of recovered humans
S_H	Number of susceptible humans
S_R	Number of susceptible reservoirs
T_H	Number of humans undergoing treatment
(1-ρ)	Proportion of exposed humans that progress to asymptomatic infected humans
α	Progression rate from exposed humans to infected humans
β	Transmission rate from the infected reservoirs, asymptomatic infected, symptomatic infected and treatment or hospitalized humans to susceptible humans.
γ	Treatment rate of symptomatic infected humans
δ	Reduction rate in transmission due to treatment or isolation of infected humans
η	Treatment rate of asymptomatic infected humans
κ	Recovery rate due to treatment of infected humans
λ	Transmission rate from the infected reservoirs to susceptible reservoirs

Λ_1	Constant recruitment rate into susceptible humans'population
Λ_2	Constant recruitment rate into susceptible reservoirs' population
μ_1	Natural death rate of the humans' population
μ_2	Natural death rate of the reservoirs' population
ν	Hunting rate of the reservoirs
ρ	Proportion of exposed humans that progress to symptomatic infected humans
ϕ	Disease-induced death rate due to Lassa fever

Local Stability Analysis of theDisease-Free Equilibrium State (DFE)

At equilibrium, $\frac{dS_H}{dt} = \frac{dE_H}{dt} = \frac{dA_H}{dt} = \frac{dI_H}{dt} = \frac{dT_H}{dt} = \frac{dR_H}{dt} = \frac{dS_R}{dt} = \frac{dI_R}{dt} = 0$ (9)

This implies,

$$\wedge_1 - \beta(I_R + I_H + A_H + \delta T_H)S_H - \mu_1 S_H = 0 \quad (10)$$

$$\beta(I_R + I_H + A_H + \delta T_H)S_H - (\alpha + \mu_1)E_H = 0 \quad (11)$$

$$(1-\rho)\alpha E_H - (\eta + \phi + \mu_1)A_H = 0 \quad (12)$$

$$\rho\alpha E_H - (\gamma + \phi + \mu_1)I_H = 0 \quad (13)$$

$$\gamma I_H + \eta A_H - (\kappa + \phi + \mu_1)T_H = 0 \quad (14)$$

$$\kappa T_H - \mu_1 R_H = 0 \quad (15)$$

$$\wedge_2 - \lambda S_R I_R - (\nu + \mu_2)S_R = 0 \quad (16)$$

$$\lambda S_R I_R - (\nu + \mu_2)I_R = 0 \quad (17)$$

Solving equations (10) through (17) simultaneously gives

$$E_0 = [S_H, E_H, A_H, I_H, T_H, R_H, S_R, I_R] = \left[\frac{\wedge_1}{\mu_1}, 0, 0, 0, 0, 0, \frac{\wedge_2}{\nu + \mu_2}, 0\right] \quad (18)$$

The Jacobean matrix of the system of equations at disease-free equilibrium state gives:

$$J(E_0)=\begin{bmatrix} -\mu_1 & 0 & -p & -p & -\delta p & 0 & 0 & -p \\ 0 & -q & p & p & \delta p & 0 & 0 & p \\ 0 & (1-\rho)\alpha & -r & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho\alpha & 0 & -s & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta & \gamma & -u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa & -\mu_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x & \dfrac{-\lambda\wedge_2}{\nu+\mu_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y \end{bmatrix} \quad (19)$$

Where,

$$\begin{bmatrix} p=\dfrac{\beta\wedge_1}{\mu_1}, q=(\alpha+\mu_1), r=(\eta+\phi+\mu_1), s=(\gamma+\phi+\mu_1), \\ u=(\kappa+\phi+\mu_1), x=(\nu+\mu_2), y=\dfrac{\lambda\wedge_2}{\nu+\mu_2}-\nu+\mu_2 \end{bmatrix} \quad (20)$$

Using Maple with elementary row operation, we transform (20) into upper triangular matrix as

$$J=\begin{bmatrix} -\mu_1 & 0 & -p & -p & -\delta p & 0 & 0 & -p \\ 0 & -q & p & p & \delta p & 0 & 0 & p \\ 0 & 0 & -\dfrac{qr-(1-\rho)p\alpha}{q} & \dfrac{(1-\rho)p\alpha}{q} & \dfrac{(1-\rho)p\alpha\delta}{q} & 0 & 0 & \dfrac{(1-\rho)p\alpha}{q} \\ 0 & 0 & 0 & -\dfrac{\rho p\alpha(r-s)+s(p\alpha-qr)}{(1-\rho)p\alpha-qr} & \dfrac{\rho p\alpha\delta r}{qr-(1-\rho)p\alpha} & 0 & 0 & \dfrac{\rho p\alpha r}{qr-(1-\rho)p\alpha} \\ 0 & 0 & 0 & 0 & -\dfrac{\begin{array}{l}\rho p\alpha\delta(\eta s-r\gamma)\\ -\eta p\alpha\delta s+\\ \rho p\alpha u(s-r)+\\ su(qr-pr)\end{array}}{\begin{array}{l}\rho p\alpha(s-r)+\\ s(qr-p\alpha)\end{array}} & 0 & 0 & \dfrac{\begin{array}{l}(\eta\rho s-r\rho\gamma\\ -\eta s)p\alpha\end{array}}{\begin{array}{l}\rho p\alpha(r-s)+\\ s(p\alpha-qr)\end{array}} \\ 0 & 0 & 0 & 0 & 0 & -\mu_1 & 0 & \dfrac{\begin{array}{l}\kappa(\eta\rho s-r\rho\gamma\\ -\eta s)p\alpha\end{array}}{\begin{array}{l}\rho p\alpha\delta(\eta s-r\gamma)\\ -\eta p\alpha\delta s+\\ \rho p\alpha u(s-r)+\\ su(qr-pr)\end{array}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -x & z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y \end{bmatrix} \quad (21)$$

The matrix in equation (21) can be written as

$$J=\begin{bmatrix} -\mu_1 & 0 & -p & -p & -\delta p & 0 & 0 & -p \\ 0 & -q & p & p & \delta p & 0 & 0 & p \\ 0 & 0 & -A_1 & \frac{(1-\rho)p\alpha}{q} & \frac{(1-\rho)p\alpha\delta}{q} & 0 & 0 & \frac{(1-\rho)p\alpha}{q} \\ 0 & 0 & 0 & -A_2 & \frac{\rho p\alpha\delta r}{qr-(1-\rho)p\alpha} & 0 & 0 & \frac{\rho p\alpha r}{qr-(1-\rho)p\alpha} \\ 0 & 0 & 0 & 0 & -A_3 & 0 & 0 & \frac{(\eta\rho s-r\rho\gamma-\eta s)p\alpha}{\rho p\alpha(r-s)+s(p\alpha-qr)} \\ 0 & 0 & 0 & 0 & 0 & -\mu_1 & 0 & \frac{\kappa(\eta\rho s-r\rho\gamma-\eta s)p\alpha}{\rho p\alpha\delta(\eta s-r\gamma)-\eta p\alpha\delta s+\rho p\alpha u(s-r)+su(qr-pr)} \\ 0 & 0 & 0 & 0 & 0 & 0 & -x & z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y \end{bmatrix} \quad (22)$$

Where ,

$$\begin{bmatrix} A_1=\frac{qr-(1-\rho)p\alpha}{q}, A_2=\frac{\rho p\alpha(r-s)+s(p\alpha-qr)}{(1-\rho)p\alpha-qr}, A_3=\frac{\rho p\alpha\delta(\eta s-r\gamma)-\eta p\alpha\delta s+\rho p\alpha u(s-r)+su(qr-pr)}{\rho p\alpha(s-r)+s(qr-p\alpha)} \end{bmatrix} \quad (23)$$

From (23), we obtain the characteristic equation,

$$\left|J-\omega I\right|$$

$$\begin{vmatrix} -\mu_1-\omega & 0 & -p & -p & -\delta p & 0 & 0 & -p \\ 0 & -q-\omega & p & p & \delta p & 0 & 0 & p \\ 0 & 0 & -A_1-\omega & \frac{(1-\rho)p\alpha}{q} & \frac{(1-\rho)p\alpha\delta}{q} & 0 & 0 & \frac{(1-\rho)p\alpha}{q} \\ 0 & 0 & 0 & -A_2-\omega & \frac{\rho p\alpha\delta r}{qr-(1-\rho)p\alpha} & 0 & 0 & \frac{\rho p\alpha r}{qr-(1-\rho)p\alpha} \\ 0 & 0 & 0 & 0 & -A_3-\omega & 0 & 0 & \frac{(\eta\rho s-r\rho\gamma}{-\eta s)p\alpha} \\ 0 & 0 & 0 & 0 & 0 & -\mu_1-\omega & 0 & \frac{\rho p\alpha(r-s)+}{s(p\alpha-qr)} \\ 0 & 0 & 0 & 0 & 0 & 0 & -x-\omega & \frac{\kappa(\eta\rho s-r\rho\gamma}{-\eta s)p\alpha} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\rho p\alpha\delta(\eta s-r\gamma)}{-\eta p\alpha\delta s+} \\ & & & & & & & \rho p\alpha u(s-r)+ \\ & & & & & & & su(qr-pr) \\ 0 & 0 & 0 & 0 & 0 & 0 & -x-\omega & z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y-\omega \end{vmatrix} = 0 \quad (24)$$

That is,

$$\left|J-\omega I\right| = (-\mu_1-\omega)(-q-\omega)(-A_1-\omega)(-A_2-\omega)(-A_3-\omega)(-\mu_1-\omega)(-x-\omega)(y-\omega) = 0 \quad (25)$$

Either ($-\mu_1-\omega=0$ or $-q-\omega=0$ or $-A_1-\omega=0$ or $-A_2-\omega=0$ or $-A_3-\omega=0$

or $-\mu_1-\omega=0$ or $-x-\omega=0$ or $y-\omega=0$) (26)

Therefore,

($\omega_1=-\mu_1$, $\omega_2=-q=-(\alpha+\mu_1)$, $\omega_3=-A_1$, $\omega_4=-A_2$, $\omega_5=-A_3$, $\omega_6=-\mu_1$

$\omega_7=-x=-(\nu+\mu_2)$, $\omega_8=y=\frac{\lambda\wedge_2}{\nu+\mu_2}-(\nu+\mu_2)$) (27)

From (27),

$\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7<0$ and $\omega_8<0$ if and only if $\frac{\lambda\wedge_2}{(\nu+\mu_2)}<(\nu+\mu_2)$ and

$\omega_8>0$ if and only if $\frac{\lambda\wedge_2}{(\nu+\mu_2)}>(\nu+\mu_2)$, Hence the disease-free equilibrium state is stable stable if $\frac{\lambda\wedge_2}{(\nu+\mu_2)}<(\nu+\mu_2)$ and unstable otherwise.

Global Stability of the Disease-Free Equilibrium State

Theorem 3.1: (**Castillo-Chavez, Feng and Huang Theorem)**

Consider epidemiological models written in the form

$$\left.\begin{aligned}\frac{dx}{dt} &= f(x,E,I)\\ \frac{dE}{dt} &= g(x.E,I)\\ \frac{dI}{dt} &= h(x,E,I)\end{aligned}\right\} \quad (28)$$

Where $x \epsilon R^r$, $E \epsilon R^s$, $I \epsilon R^n$, r, s, n ≥ 0. The components of x represent the classes of susceptible, recovered and other non-infected classes. The components of E represent exposed and latent classes and the components I represent infected and infectious classes.

Let equation (28) be written in the form

$$\left.\begin{aligned}\frac{dx}{dt} &= F(x,I)\\ \frac{dI}{dt} &= G(x,I), G(x,0)=0\end{aligned}\right\} \quad (29)$$

Where $x \epsilon R^m$ denotes uninfected classes and $I \epsilon R^n$ denotes infected classes including latent and exposed, and infectious classes.

Then;

The disease-free-equilibrium state $U_0(x^*,0)$ is globally asymptotically stable provided $R_0<1$ and the following two conditions (H1) and (H2) are satisfied.

(H1) For $\frac{dx}{dt} = F(x,0), x^*$ is globally asymptotically stable

(H2) $G(x,I) = AI - \overline{G}(x,I), \overline{G}(x,I) \geq 0$ for $(x,I) \epsilon D$

Where $A=G(x^*, 0)$ is an M-matrix(the off diagonal elements are nonnegative) and D is the region where the model makes biological sense.

Proof

in this study, the global stability of the disease-free-equilibrium is established using the two conditions (H1) and (H2) as stated in (Castillo-Chavez et al, 2001) must be satisfied for $R_0<1$.

For the first condition, We write our equations of the model (10) through (17) in the form

$$\frac{dX}{dt} = F(X,Y) \tag{30}$$

$$\frac{dY}{dt} = G(X,Y); G(X,0) = 0 \tag{31}$$

Where $X = (S_H, R_H, S_R)$ and $Y = (E_H, A_H, I_H, T_H, I_R)$

With the elements $X \epsilon R^3$ representing the uninfected compartments and the elements $Y \epsilon R^5$ representing infected compartments.

From equation (18) we have our disease-free-equilibrium as

$$U_0(X^*, 0) = [S_H, E_H, A_H, I_H, T_H, R_H, S_R, I_R] = \left[\frac{\wedge_1}{\mu_1}, 0, 0, 0, 0, 0, \frac{\wedge_2}{\nu + \mu_2}, 0\right] \tag{32}$$

From (30), we have

$$\frac{dX}{dt} = F(X,0) = \begin{bmatrix} \wedge_1 - \mu_1 S_H \\ 0 \\ \wedge_2 - (\nu + \mu_2) S_R \end{bmatrix} \tag{33}$$

From equation (33), we have

$$\frac{dS_H}{dt} = \wedge_1 - \mu_1 S_H \tag{34}$$

Equation (34) can be written as

$$\frac{dS_H}{dt} + \mu_1 S_H = \wedge_1 \tag{35}$$

The integrating factor (IF) of (35) is $e^{\mu_1 t}$ (36)

Multiplying both sides of equation (35) by (36) gives

$$e^{\mu_1 t}\frac{dS_H}{dt} + \mu_1 e^{\mu_1 t} S_H = \wedge_1 e^{\mu_1 t} \tag{37}$$

Equation (37) can be written as

$$\frac{d}{dt}(S_H e^{\mu_1 t}) = \wedge_1 e^{\mu_1 t} \tag{38}$$

Integrating both sides gives

$$S_H e^{\mu_1 t} = \wedge_1 \int_0^t e^{\mu_1 \tau} d\tau + c \tag{39}$$

$$S_H e^{\mu_1 t} = \frac{\wedge_1}{\mu_1} e^{\mu_1 t} + c \tag{40}$$

$$\Rightarrow S_H(t) = \frac{\wedge_1}{\mu_1} + ce^{-\mu_1 t} \tag{41}$$

From equation (41), for $S_H(0)=S_{H0}$, we have

$$c = S_{H0} - \frac{\wedge_1}{\mu_1} \tag{42}$$

Substituting equation (42) into (41) gives

$$S_H(t) = \frac{\wedge_1}{\mu_1}\left(1 - e^{-\mu_1 t}\right) + S_{H0} e^{-\mu_1 t} \tag{43}$$

As $t \to \infty$, $S_H(t) \to \frac{\wedge_1}{\mu_1}$ (44)

Similarly, from equation (33), we have

$$\frac{dS_R}{dt} = \wedge_2 - (\nu + \mu_2) S_R \tag{45}$$

Equation (45) can be written as

$$\frac{dS_R}{dt} + (\nu + \mu_2) S_R = \wedge_2 \tag{46}$$

The integrating factor of (46) is $e^{(\nu+\mu_2)t}$ (47)

Multiplying both sides of equation (46) by (47) gives

$$e^{(\nu+\mu_2)t} \frac{dS_R}{dt} + (\nu + \mu_2) e^{(\nu+\mu_2)t} S_R = \wedge_2 e^{(\nu+\mu_2)t} \tag{48}$$

Equation (48) can be written as

$$\frac{d}{dt}(S_R e^{(\nu+\mu_2)t}) = \wedge_2 e^{(\nu+\mu_2)t} \tag{49}$$

Integrating both sides gives

$$\frac{d}{dt}(S_R e^{(\nu+\mu_2)t}) = \wedge_2 \int_0^t e^{(\nu+\mu_2)\tau} d\tau + k \tag{50}$$

$$(S_R e^{(\nu+\mu_2)t}) = \frac{\wedge_2}{(\nu + \mu_2)} e^{(\nu+\mu_2)t} + k \tag{51}$$

Equation (51) can be written as

$$S_R(t) = \frac{\wedge_2}{(\nu + \mu_2)} + ke^{-(\nu+\mu_2)t} \tag{52}$$

From equation (52), for $S_R(0)=S_{R0}$, we have

$$k = S_{R0} - \frac{\wedge_2}{\nu + \mu_2} \tag{53}$$

Substituting equation (53) into (52) gives

$$S_R(t) = \frac{\wedge_2}{(\nu + \mu_2)}(1 - e^{-(\nu+\mu_2)t}) + S_{R0}e^{-(\nu+\mu_2)t} \tag{54}$$

As $t \to \infty$, $S_R(t) \to \frac{\wedge_2}{(\nu + \mu_2)}$ (55)

From equations (44) and (55), it implies that regardless of the values of $S_H(0)$ and $S_R(0)$, $S_H(t) \to \frac{\wedge_1}{\mu_1}$ and $S_R(t) \to \frac{\wedge_2}{(\nu + \mu_2)}$ as $t \to \infty$.

For the second condition (H2), $G(X,Y) = AI - \overline{G}(X,Y)$, we have

$$A = \begin{bmatrix} -(\alpha+\mu_1) & \beta & \beta & \beta\delta & \beta \\ (1-\rho)\alpha & -(\eta+\phi+\mu_1) & 0 & 0 & 0 \\ \rho\alpha & 0 & -(\gamma+\phi+\mu_1) & 0 & 0 \\ 0 & \eta & \gamma & -(\kappa+\phi+\mu_1) & 0 \\ 0 & 0 & 0 & 0 & \lambda-(\nu+\mu_2) \end{bmatrix} \tag{56}$$

$$\overline{G}(X,Y) = \begin{bmatrix} \beta(A_H + I_H + \delta T_H + I_R)(1-S_H) \\ 0 \\ 0 \\ 0 \\ \lambda I_R(1-S_R) \end{bmatrix} \tag{57}$$

Clearly, from equation (56), A is an M-matrix and from equation (57), $\overline{G}(X,Y) \geq 0$,

Hence $U_0(X^*,0) = \left[\frac{\wedge_1}{\mu_1}, 0, 0, 0, 0, 0, \frac{\wedge_2}{\nu+\mu_2}, 0\right]$ is globally asymptotically stable.

Basic Reproduction Number

The basic reproductive number, R_0, is defined as the number of secondary infections that an infective individual produces over the duration of the infectious period in an entirely susceptible population. The basic reproduction number is a threshold number that if it is less than unity, that is if $R_0<1$ then the disease-free-equilibrium (DFE) is locally asymptotically stable, and if it is greater than unity,

that is if $R_0>1$ then the disease-free-equilibrium is unstable. In this study, we employ the next generation matrix approach as described by (Driessche and Wathmough, 2008) to obtain our Basic Reproductive Number. We take the basic reproduction number as the spectral radius of the product of the two matrices, F and V^{-1}, that is, $R_0=\rho(FV^{-1})$.

Our model has four infected classes for the human population; hence we have the next generation matrices F and V for new infection terms and transmission terms respectively as

$$F=\begin{bmatrix} 0 & \frac{\beta\wedge_1}{\mu_1} & \frac{\beta\wedge_1}{\mu_1} & \frac{\beta\delta\wedge_1}{\mu_1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{58}$$

$$V=\begin{bmatrix} \alpha+\mu_1 & 0 & 0 & 0 \\ -(1-\rho)\alpha & \eta+\phi+\mu_1 & 0 & 0 \\ -\rho\alpha & 0 & \gamma+\phi+\mu_1 & 0 \\ 0 & -\eta & -\gamma & \kappa+\phi+\mu_1 \end{bmatrix} \tag{59}$$

Using Maple software, we have

$$V^{-1}=\begin{bmatrix} \frac{1}{\alpha+\mu_1} & 0 & 0 & 0 \\ \frac{(1-\rho)\alpha}{(\alpha+\mu_1)(\eta+\phi+\mu_1)} & \frac{1}{\eta+\phi+\mu_1} & 0 & 0 \\ \frac{\rho\alpha}{(\alpha+\mu_1)(\gamma+\phi+\mu_1)} & 0 & \frac{1}{\gamma+\phi+\mu_1} & 0 \\ \frac{\alpha(-\eta\phi\rho-\eta\rho\mu_1+\phi\rho\gamma+\rho\gamma\mu_1+\eta\phi+\eta\gamma+\eta\mu_1)}{(\eta+\phi+\mu_1)(\alpha+\mu_1)(\gamma+\phi+\mu_1)(\kappa+\phi+\mu_1)} & \frac{\eta}{(\eta+\phi+\mu_1)(\kappa+\phi+\mu_1)} & \frac{\gamma}{(\gamma+\phi+\mu_1)(\kappa+\phi+\mu_1)} & \frac{1}{(\kappa+\phi+\mu_1)} \end{bmatrix} \tag{60}$$

$$FV^{-1}=\begin{bmatrix} \dfrac{\beta\wedge_1(1-\rho)\alpha}{\mu_1(\alpha+\mu_1)(\eta+\phi+\mu_1)}+\dfrac{\beta\wedge_1\rho\alpha}{\mu_1(\alpha+\mu_1)(\gamma+\phi+\mu_1)} & \left(\dfrac{\beta\wedge_1}{\mu_1(\eta+\phi+\mu_1)}+\dfrac{\beta\wedge_1\delta\eta}{\mu_1(\eta+\phi+\mu_1)(\kappa+\phi+\mu_1)}\right) & \left(\dfrac{\beta\wedge_1}{\mu_1(\gamma+\phi+\mu_1)}+\dfrac{\beta\wedge_1\delta\gamma}{\mu_1(\gamma+\phi+\mu_1)(\kappa+\phi+\mu_1)}\right) & \dfrac{\beta\wedge_1\delta}{\mu_1(\kappa+\phi+\mu_1)} \\ \beta\wedge_1\delta\alpha\dfrac{\begin{pmatrix}\eta(1-\rho)\\(\gamma+\phi+\mu_1)\\+\gamma\rho(\eta+\phi+\mu_1)\end{pmatrix}}{\begin{pmatrix}\mu_1(\alpha+\mu_1)\\(\eta+\phi+\mu_1)\\(\gamma+\phi+\mu_1)\\(\kappa+\phi+\mu_1)\end{pmatrix}} & & & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{61}$$

The characteristics equation is given by

$$\left|FV^{-1}-\omega I\right|=\begin{vmatrix} \dfrac{\beta\wedge_1(1-\rho)\alpha}{\mu_1(\alpha+\mu_1)(\eta+\phi+\mu_1)}+\dfrac{\beta\wedge_1\rho\alpha}{\mu_1(\alpha+\mu_1)(\gamma+\phi+\mu_1)} & \left(\dfrac{\beta\wedge_1}{\mu_1(\eta+\phi+\mu_1)}+\dfrac{\beta\wedge_1\delta\eta}{\mu_1(\eta+\phi+\mu_1)(\kappa+\phi+\mu_1)}\right) & \left(\dfrac{\beta\wedge_1}{\mu_1(\gamma+\phi+\mu_1)}+\dfrac{\beta\wedge_1\delta\gamma}{\mu_1(\gamma+\phi+\mu_1)(\kappa+\phi+\mu_1)}\right) & \dfrac{\beta\wedge_1\delta}{\mu_1(\kappa+\phi+\mu_1)} \\ \beta\wedge_1\delta\alpha\dfrac{\begin{pmatrix}\eta(1-\rho)\\(\gamma+\phi+\mu_1)\\+\gamma\rho\\(\eta+\phi+\mu_1)\end{pmatrix}}{\begin{pmatrix}\mu_1(\alpha+\mu_1)\\(\eta+\phi+\mu_1)\\(\gamma+\phi+\mu_1)\\(\kappa+\phi+\mu_1)\end{pmatrix}} & & & \\ -\omega & & & \\ 0 & -\omega & 0 & 0 \\ 0 & 0 & -\omega & 0 \\ 0 & 0 & 0 & -\omega \end{vmatrix}=0 \tag{62}$$

$$\left|FV^{-1}-\omega I\right|=\left(\frac{\beta\wedge_1(1-\rho)\alpha}{\mu_1(\alpha+\mu_1)(\eta+\phi+\mu_1)}+\frac{\beta\wedge_1\rho\alpha}{\mu_1(\alpha+\mu_1)(\gamma+\phi+\mu_1)}+\frac{\beta\wedge_1\delta\alpha\begin{pmatrix}\eta(1-\rho)\\(\gamma+\phi+\mu_1)\\+\gamma\rho\\(\eta+\phi+\mu_1)\end{pmatrix}}{\begin{pmatrix}\mu_1(\alpha+\mu_1)\\(\eta+\phi+\mu_1)\\(\gamma+\phi+\mu_1)\\(\kappa+\phi+\mu_1)\end{pmatrix}}-\omega\right)(-\omega)(-\omega)(-\omega)=0 \tag{63}$$

Therefore, the Eigenvalues are

$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix}=\begin{bmatrix}\dfrac{\beta\wedge_1(1-\rho)\alpha}{\mu_1(\alpha+\mu_1)(\eta+\phi+\mu_1)}+\dfrac{\beta\wedge_1\rho\alpha}{\mu_1(\alpha+\mu_1)(\gamma+\phi+\mu_1)}+\dfrac{\beta\wedge_1\delta\alpha\begin{pmatrix}\eta(1-\rho)\\(\gamma+\phi+\mu_1)\\+\gamma\rho(\eta+\phi+\mu_1)\end{pmatrix}}{\begin{pmatrix}\mu_1(\alpha+\mu_1)(\eta+\phi+\mu_1)\\(\gamma+\phi+\mu_1)(\kappa+\phi+\mu_1)\end{pmatrix}}\\ 0\\ 0\\ 0\end{bmatrix} \quad (64)$$

Hence,

$$R_{0,H}=\frac{\beta\wedge_1(1-\rho)\alpha}{\mu_1(\alpha+\mu_1)(\eta+\phi+\mu_1)}+\frac{\beta\wedge_1\rho\alpha}{\mu_1(\alpha+\mu_1)(\gamma+\phi+\mu_1)}+\frac{\beta\wedge_1\delta\alpha\begin{pmatrix}\eta(1-\rho)\\(\gamma+\phi+\mu_1)\\+\gamma\rho(\eta+\phi+\mu_1)\end{pmatrix}}{\begin{pmatrix}\mu_1(\alpha+\mu_1)(\eta+\phi+\mu_1)\\(\gamma+\phi+\mu_1)(\kappa+\phi+\mu_1)\end{pmatrix}} \quad (65)$$

And for the reservoir population, our model has one infected class; hence we have the next generation matrices F and V for new infection terms and transmission terms respectively as

$$F=\left[\frac{\lambda\wedge_2}{\nu+\mu_2}\right], \qquad V=\nu+\mu_2 \quad (66)$$

Therefore,

$$R_{0R}=\frac{\lambda\wedge_2}{(\nu+\mu_2)^2} \quad (67)$$

Sensitivity Analysis

Sensitivity analysis confirms how significant each parameter is to disease transmission. Sensitivity of each parameter is inspected with respect to the basic reproduction number. In this way, we can discover the parameters that are more sensitive to the disease and by decreasing the parameters will also reduce the transmission of the disease. This fact is vital to data assimilation and complex nonlinear model reduction as much as it is to experimental design (Poweell *et al*, 2005). Transmission of disease greatly depends on the basic reproduction number R_0 (Mohammed et al, 2015). Sensitivity analysis is used to ascertain parameters that have huge effect on the R_0 and should be targeted by intervention strategies (Onuorah *et al, 2016).*

Sensitivity index allows us to determine the relative changes in a variable with respect to changes in a parameter. The normalized sensitivity index of a parameter R_0 with respect to a parameter q when R_0 is a differential function of q is defined as:

$$\varepsilon_q^{R_0} = \frac{\partial R_0}{\partial q} \times \frac{q}{R_0} \quad (68)$$

Recall from equations (65) and (67) that $R_{0,H}$ and $R_{0,R}$ are respectively;

$$R_{0,H} = \frac{\beta \wedge_1 (1-\rho)\alpha}{\mu_1(\alpha+\mu_1)(\eta+\phi+\mu_1)} + \frac{\beta \wedge_1 \rho\alpha}{\mu_1(\alpha+\mu_1)(\gamma+\phi+\mu_1)} + \frac{\beta \wedge_1 \delta\alpha \begin{pmatrix} \eta(1-\rho) \\ (\gamma+\phi+\mu_1) \\ +\gamma\rho(\eta+\phi+\mu_1) \end{pmatrix}}{\begin{pmatrix} \mu_1(\alpha+\mu_1)(\eta+\phi+\mu_1) \\ (\gamma+\phi+\mu_1)(\kappa+\phi+\mu_1) \end{pmatrix}}$$

$$R_{0R} = \frac{\lambda \wedge_2}{(\nu+\mu_2)^2}$$

Table 1; Initial conditions and parameter values

Parameters and State Variables	Value	Source
$S_H(0)$	10000	Assumed
$E_H(0)$	7000	Assumed
$A_H(0)$	5600	Calculated
$I_H(0)$	1400	Calculated
$T_H(0)$	6500	Assumed
$R_H(0)$	5500	Assumed
$S_R(0)$	3000	Assumed
$I_R(0)$	700	Assumed
$\wedge_1$	1200	Assumed
$\wedge_2$	400	Assumed
μ_1	0.000047	Mohammed et. al. (2015)
μ_2	0.08	Assumed
β	0.002	Assumed
λ	0.03	Assumed
δ	0.2	Assumed
α	0.05	Assumed

ρ	0.2	WHO (2017)
ϕ	0.01	WHO (2017)
η	0.5	Assumed
γ	0.8	Assumed
κ	0.8	Assumed
ν	0.02	Assumed

From the explicit formulas for $R_{0,H}$ and $R_{0,R}$, the analytical expressions for the basic reproduction numbers with respect to the parameters they contained are obtained using equation (68), the values in Table 1 and Maple software as follows;

$$\varepsilon_{\wedge_1}^{R_{0,H}} = \frac{\partial R_{0,H}}{\partial \wedge_1} \times \frac{\wedge_1}{R_{0,H}} = 1.000000000 \quad (69)$$

$$\varepsilon_{\alpha}^{R_{0,H}} = \frac{\partial R_{0,H}}{\partial \alpha} \times \frac{\alpha}{R_{0,H}} = 0.2857142861 \quad (70)$$

$$\varepsilon_{\beta}^{R_{0,H}} = \frac{\partial R_{0,H}}{\partial \beta} \times \frac{\beta}{R_{0,H}} = 1.000000000 \quad (71)$$

$$\varepsilon_{\delta}^{R_{0,H}} = \frac{\partial R_{0,H}}{\partial \delta} \times \frac{\delta}{R_{0,H}} = 0.1153835507 \quad (72)$$

$$\varepsilon_{\eta}^{R_{0,H}} = \frac{\partial R_{0,H}}{\partial \eta} \times \frac{\eta}{R_{0,H}} = -0.7144562164 \quad (73)$$

$$\varepsilon_{\gamma}^{R_{0,H}} = \frac{\partial R_{0,H}}{\partial \gamma} \times \frac{\gamma}{R_{0,H}} = -0.1165282340 \quad (74)$$

$$\varepsilon_{\kappa}^{R_{0,H}} = \frac{\partial R_{0,H}}{\partial \kappa} \times \frac{\kappa}{R_{0,H}} = 0.0003773505709 \quad (75)$$

$$\varepsilon_{\mu_1}^{R_{0,H}} = \frac{\partial R_{0,H}}{\partial \mu_1} \times \frac{\mu_1}{R_{0,H}} = -0.924249279 \quad (76)$$

$$\varepsilon_{\phi}^{R_{0,H}} = \frac{\partial R_{0,H}}{\partial \phi} \times \frac{\phi}{R_{0,H}} = -0.01926749642 \quad (77)$$

$$\varepsilon_{\rho}^{R_{0,H}} = \frac{\partial R_{0,H}}{\partial \rho} \times \frac{\rho}{R_{0,H}} = -0.06843285452 \quad (78)$$

$$\varepsilon_{\lambda}^{R_{0,R}} = \frac{\partial R_{0,R}}{\partial \lambda} \times \frac{\lambda}{R_{0,R}} = 1.000000000 \quad (79)$$

$$\varepsilon_{\wedge_2}^{R_{0,R}} = \frac{\partial R_{0,R}}{\partial \wedge_2} \times \frac{\wedge_2}{R_{0,R}} = 1.000000000 \quad (80)$$

$$\varepsilon_{\mu_2}^{R_{0,R}} = \frac{\partial R_{0,R}}{\partial \mu_2} \times \frac{\mu_2}{R_{0,R}} = -0.5714285716 \quad (81)$$

$$\varepsilon_{\nu}^{R_{0,R}} = \frac{\partial R_{0,R}}{\partial \nu} \times \frac{\nu}{R_{0,R}} = -0.4000000000 \quad (82)$$

Table 2; Sensitivity Indices of Parameters of Humans' Basic Reproduction Number ($R_{0,H}$)

Parameter	Sign	Sensitivity Index
Λ_1	+	1.0000000000
α	+	0.2857142861
β	+	1.0000000000
δ	+	0.1153835507
η	-	0.7144562164
γ	-	0.1165282340
κ	+	0.0003773506
μ_1	-	0.924249279
ϕ	-	0.0192674964
ρ	-	0.0684328545

Table 3; Sensitivity Indices of the Parameters of the Reservoirs' Basic Reproduction Number ($R_{0,R}$)

Parameter	Sign	Sensitivity Index
Λ_2	+	1.0000000000
λ	+	1.0000000000
ν	-	0.4000000000
μ_2	-	0.5714285716

Conclusion.

In this study, we carry out the local stability analysis of the disease free equilibrium state and sensitivity analysis of parameters with respect to the basic reproductive number of the model presented in (Enagi and Muhammed, 2019). The disease-free equilibrium state is stable if $\frac{\lambda \wedge_2}{(\nu + \mu_2)} < (\nu + \mu_2)$ and $R_0 \epsilon (R_{0H}, R_{0R}) < 1$ otherwise unstable. We also analysed the global stability of the disease-free-equilibrium using Castillo-Chavez, Feng and Huang approach. The result shows that the disease-free-equilibrium state is globally asymptotically stable. We carried out sensitivity analysis on both the humans' and Rodents' basic reproduction numbers $R_{0,H}$ and $R_{0,R}$ respectively, and it was ascertained that the most sensitive parameters to the humans' basic reproduction number are the natural birth rate, Λ_1 and contact rate β, while the most sensitive parameters to the rodents' basic reproduction number are the natural birth rate, Λ_2 and contact rate, λ. Hence any change in these parameters greatly affects the dynamics of the disease.

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