AN EFFICIENT METHOD FOR SOLVING FIRST ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract:
This paper considered the hybrid numerical method for the solution of first order delay differential equations. The continuous formulation of the method was derived through collocation method by matrix inversion technique. The convergence analysis of the method was carried out in which found to be both consistent and zero-stable. The region of P and Q-stability were plotted. The method was implemented in block form which tested the efficiency of the method on some numerical examples.

Keywords: *Efficient method, continuous formulations, delay differential equations, P-stabilty, Q-stability and Multi-step Method.*

INTRODUCTION

Delay Differential Equations (DDEs) are those in which the time evolution of the state variable can depend on the history or past values in some way at discrete times. In this paper we focus on a specific type of these equations of the form:

$$\begin{aligned} y'(t) &= f(\mathrm{t}, \mathrm{y}, y(t-\tau)), \quad && t > t_0, \tau > 0 \\ y(t) &= \varphi(t) && t \le t_0 \end{aligned} \tag{1}$$

where $\varphi(t)$ is the initial function, $\tau(t, y(t))$ is called a delay, $t-\tau(t, y(t))$ is the delay argument and $y(t-\tau(t, y(t)))$ is the solution of the delay argument. The delay can be constant $\tau = C$, time delay $\tau = \tau(t)$ or state delay $\tau = \tau(t, y(t))$.

Delay differential equations have been widely used in many science areas. Delays are usually involved in Biology (heredity in population dynamics and epidemiology), Chemistry (behaviors in chemical kinetics), Economics

(dynamics of business cycles), Physics (hydraulic networks electric and pneumatic) and other branches.
Different methods have been proposed by some researchers to solve first order DDEs. At time when theoretical solutions for delay differential equations are difficult or can not to be find, then mathematicians or engineers resort to approximate solutions that can be made as accurately as possible. Many numerical methods for solving ordinary differential equations (ODEs) have been extended to solve DDEs with some modifications. The methods include Runge-Kutta and its family methods and linear multistep methods. All of these methods give only one numerical solution in a given step as in Oberle and Pesh (1981) Suleiman and Ishak (2010), Enright and Hayashi (1997), Ismail et.al (2002) and Jackiewicz and Lo (2006). Another method that produce more than one numerical solution in a given step is called a block method. These methods are efficient since the number functions evaluation can be reduced, like in Ishak et.al (2008), Ishak et.al (2010), Majid and Suleiman (2011), San et.al (2011) and Tian et.al (2009). Recently, the methods that gained more efficiency is a hybrid methods in which the order of the methods will be increases without an increase in the step number, as such the accuracy of the methods will also be increase. In this paper the hybrid reformulated block backward differentiation methods will be used to solve first order delay differential equations and an efficient formula in Sirisena and Yakubu (2019) will be implemented for the solution of delay argument.

METHOD

In this section, we shall consider the derivation of the continuous formulations of the method. Its continuous forms shall be obtained through multistep collocation method using the matrix inversion technique of Sirisena (1997), and we deduced the discrete schemes from it. The order, error constants, zero stability, consistency and convergence of the discrete schemes and their regions of absolute stability were studied and plotted.

DERIVATION OF MULTISTEP COLLOCATION METHOD

In Sirisena (1997) a *k*-step multistep collocation method with *m* collocation points was obtained as

$$y(x)=\sum_{j=0}^{t-1}\alpha_j(x)y_{n+j}+h\sum_{j=0}^{m-1}\beta_j(x)f(x,y(x))$$

(2) where $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients of the method defined as

$$\alpha_j(x) = \sum_{i=0}^{t+m-1} \alpha_{j,i+1} x^i \quad \text{for } j = \{0,1,\ldots,t-1\} \tag{3}$$

$$h\beta_j(x) = \sum_{i=0}^{t+m-1} h\beta_{j,i+1} x^i \quad \text{for } j = \{0,1,\ldots,m-1\} \tag{4}$$

where $x_0, \ldots, x_{m-1}$ are the m collocation points and x_{n+j}, $j = 0, 1, 2, \ldots, t-1$ are the t arbitrarily chosen interpolation points.

To get $\alpha_j(x)$ and $\beta_j(x)$, Sirisena (1997) arrived at a matrix equation of the form

$$DC = I \tag{5}$$

Where I is the identity matrix of dimension $(t+m)\times(t+m)$ while D and C are matrices defined as

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2x_0 & \cdots & (t+m-1)x_0^{t+m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{m-1} & \cdots & (t+m-1)x_{m-1}^{t+m-2} \end{bmatrix} \tag{6}$$

$$C = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{t-1,1} & h\beta_{0,1} & \cdots & h\beta_{m-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{t-1,2} & h\beta_{0,2} & \cdots & h\beta_{m-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,t+m} & \alpha_{1,t+m} & \cdots & \alpha_{t-1,t+m} & h\beta_{0,t+m} & \cdots & h\beta_{m-1,t+m} \end{bmatrix} \tag{7}$$

It follows from (5) that the columns of $C = D^{-1}$ give the continuous coefficients of the continuous scheme (2).

Subsequently, the continuous formulations of the method shall be derived.

DERIVATION OF CONTINUOUS FORMULATION OF THE METHOD

Here, also the number of interpolation points, $t = 1$ and the number of collocation points, m = 4. Therefore, (2) becomes:

$$y(x) = \alpha_3(x)y_{n+3} + h[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3} + \beta_4(x)f_{n+4}] \tag{8}$$

Also the matrix D in (3.6) becomes

$$D = \begin{bmatrix} 1 & x_n+3h & (x_n+3h)^2 & (x_n+3h)^3 & (x_n+3h)^4 & (x_n+3h)^5 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 \\ 0 & 1 & 2x_n+2h & 3(x_n+h)^2 & 4(x_n+h)^3 & 5(x_n+h)^4 \\ 0 & 1 & 2x_n+4h & 3(x_n+2h)^2 & 4(x_n+2h)^3 & 5(x_n+2h)^4 \\ 0 & 1 & 2x_n+6h & 3(x_n+3h)^2 & 4(x_n+3h)^3 & 5(x_n+3h)^4 \\ 0 & 1 & 2x_n+8h & 3(x_n+4h)^2 & 4(x_n+4h)^3 & 5(x_n+4h)^4 \end{bmatrix} \qquad (9)$$

The inverse of the matrix $C = (D^{-1})$ is computed using Maple 17 from which the following continuous scheme is also obtained using (2)

$$\begin{aligned}
(x) = y_{n+3} + \Bigg(& -\frac{1}{720}\frac{(x_n+3h)(81h^4+213h^3x_n+179h^2x_n^2+57hx_n^3+6x_n^4)}{h^4} \\
& + \frac{1}{24}\frac{x(24h^4+50h^3x_n+35h^2x_n^2+10hx_n^3+x_n^4)}{h^4} \\
& - \frac{1}{24}\frac{x^2(25h^3+35h^2x_n+15hx_n^2+2x_n^3)}{h^4} + \frac{1}{72}\frac{x^3(35h^2+30hx_n+6x_n^2)}{h^4} \\
& - \frac{1}{48}\frac{x^4(5h+2x_n)}{h^4} + \frac{1}{120}\frac{x^5}{h^4}\Bigg) f_n + \Bigg(\\
& -\frac{1}{360}\frac{(x_n+3h)(153h^4-51h^3x_n-223h^2x_n^2-99hx_n^3-12x_n^4)}{h^4} \\
& - \frac{1}{6}\frac{xx_n(24h^3+26h^2x_n+9hx_n^2+x_n^3)}{h^4} \\
& + \frac{1}{12}\frac{x^2(24h^3+52h^2x_n+27hx_n^2+4x_n^3)}{h^4} - \frac{1}{18}\frac{x^3(26h^2+27hx_n+6x_n^2)}{h^4} \\
& + \frac{1}{24}\frac{x^4(9h+4x_n)}{h^4} - \frac{1}{30}\frac{x^5}{h^4}\Bigg) f_{n+1} + \Bigg(\\
& -\frac{1}{60}\frac{(x_n+3h)(18h^4-6h^3x_n+32h^2x_n^2+21hx_n^3+3x_n^4)}{h^4} \\
& + \frac{1}{4}\frac{xx_n(12h^3+19h^2x_n+8hx_n^2+x_n^3)}{h^4} - \frac{1}{4}\frac{x^2(6h^3+19h^2x_n+12hx_n^2+2x_n^3)}{h^4} \\
& + \frac{1}{12}\frac{x^3(19h^2+24hx_n+6x_n^2)}{h^4} - \frac{1}{4}\frac{x^4(x_n+2h)}{h^4} + \frac{1}{20}\frac{x^5}{h^4}\Bigg) f_{n+2} + \Bigg(\\
& -\frac{1}{360}\frac{(x_n+3h)(63h^4-21h^3x_n-73h^2x_n^2-69hx_n^3-12x_n^4)}{h^4} \\
& - \frac{1}{6}\frac{xx_n(8h^3+14h^2x_n+7hx_n^2+x_n^3)}{h^4} + \frac{1}{12}\frac{x^2(8h^3+28h^2x_n+21hx_n^2+4x_n^3)}{h^4} \\
& - \frac{1}{18}\frac{x^3(14h^2+21hx_n+6x_n^2)}{h^4} + \frac{1}{24}\frac{x^4(7h+4x_n)}{h^4} - \frac{1}{30}\frac{x^5}{h^4}\Bigg) f_{n+3} \\
& + \Bigg(\frac{1}{720}\frac{(x_n+3h)(9h^4-3h^3x_n-29h^2x_n^2-27hx_n^3-6x_n^4)}{h^4} \\
& + \frac{1}{24}\frac{xx_n(6h^3+11h^2x_n+6hx_n^2+x_n^3)}{h^4} - \frac{1}{24}\frac{x^2(3h^3+11h^2x_n+9hx_n^2+2x_n^3)}{h^4} \\
& + \frac{1}{72}\frac{x^3(11h^2+18hx_n+6x_n^2)}{h^4} - \frac{1}{48}\frac{x^4(3h+2x_n)}{h^4} + \frac{1}{120}\frac{x^5}{h^4}\Bigg) f_{n+4}
\end{aligned} \qquad (10)$$

Evaluating and simplifying (10) at $x = x_n$, $x = x_{n+1}, x = x_{n+2}$ and $x = x_{n+4}$, the following discrete schemes are obtained:

$$
\begin{aligned}
y_n &= y_{n+3} - \frac{27}{80} hf_n - \frac{51}{40} hf_{n+1} - \frac{9}{10} hf_{n+2} - \frac{21}{40} hf_{n+3} + \frac{3}{80} hf_{n+4} \\
y_{n+1} &= y_{n+3} + \frac{1}{90} hf_n - \frac{17}{45} hf_{n+1} - \frac{19}{15} hf_{n+2} - \frac{17}{45} hf_{n+3} + \frac{1}{90} hf_{n+4} \\
y_{n+2} &= y_{n+3} - \frac{11}{720} hf_n + \frac{37}{360} hf_{n+1} - \frac{19}{30} hf_{n+2} - \frac{173}{360} hf_{n+3} + \frac{19}{720} hf_{n+4} \\
y_{n+4} &= y_{n+3} - \frac{19}{720} hf_n + \frac{53}{360} hf_{n+1} - \frac{11}{30} hf_{n+2} + \frac{323}{360} hf_{n+3} + \frac{251}{720} hf_{n+4}
\end{aligned}
\qquad (11)
$$

CONVERGENCE ANALYSIS

In this section the order, error constant, consistency and zero stability of the derived discrete schemes shall be considered

ORDER AND ERROR CONSTANT

Here the order and error constants of the discrete schemes in (11) are found in block form as follows:

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 - \beta_0 - \beta_1 - \beta_2 - \beta_3 - \beta_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_2 = \frac{1}{2}\alpha_1 + 2\alpha_2 + \frac{9}{2}\alpha_3 + 8\alpha_4 - \beta_1 - 2\beta_2 - 3\beta_3 - 4\beta_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_3 = \frac{1}{6}\alpha_1 + \frac{4}{3}\alpha_2 + \frac{9}{2}\alpha_3 + \frac{32}{3}\alpha_4 - \frac{1}{2}\beta_1 - 2\beta_2 - \frac{9}{2}\beta_3 - 8\beta_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_4 = \frac{1}{24}\alpha_1 + \frac{2}{3}\alpha_2 + \frac{27}{8}\alpha_3 + \frac{32}{3}\alpha_4 - \frac{1}{6}\beta_1 - \frac{4}{3}\beta_2 - \frac{9}{2}\beta_3 - \frac{32}{3}\beta_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_5 = \frac{1}{120}\alpha_1 + \frac{4}{15}\alpha_2 + \frac{81}{40}\alpha_3 + \frac{128}{15}\alpha_4 - \frac{1}{24}\beta_1 - \frac{2}{3}\beta_2 - \frac{27}{8}\beta_3 - \frac{32}{3}\beta_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_6 = \begin{bmatrix} -\frac{3}{160} \\ 0 \\ -\frac{11}{1440} \\ -\frac{3}{160} \end{bmatrix}$$

Therefore, (11) has order, $p = 5$ and error constants $= -\frac{3}{160}, 0, -\frac{11}{1440}, -\frac{3}{160}$

CONSISTENCY

All the schemes in (11) have their orders greater than one, hence the schemes are consistent.

ZERO STABILITY

The zero stability of the discrete schemes in (11) is determined in block form as follows

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}$$

$$+h \begin{pmatrix} -\frac{51}{40} & -\frac{9}{10} & -\frac{21}{40} & \frac{3}{80} \\ -\frac{17}{45} & -\frac{19}{15} & -\frac{17}{45} & \frac{1}{90} \\ \frac{37}{360} & -\frac{19}{30} & -\frac{173}{360} & \frac{19}{720} \\ \frac{53}{360} & -\frac{11}{30} & \frac{323}{360} & \frac{251}{720} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & -\frac{27}{80} \\ 0 & 0 & 0 & \frac{1}{90} \\ 0 & 0 & 0 & -\frac{11}{720} \\ 0 & 0 & 0 & -\frac{19}{720} \end{pmatrix} \begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}$$

where $A_2^{(2)} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, A_1^{(2)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

and $B_2^{(2)} = \begin{pmatrix} -\frac{51}{40} & -\frac{9}{10} & -\frac{21}{40} & \frac{3}{80} \\ -\frac{17}{45} & -\frac{19}{15} & -\frac{17}{45} & \frac{1}{90} \\ \frac{37}{360} & -\frac{19}{30} & -\frac{173}{360} & \frac{19}{720} \\ \frac{53}{360} & -\frac{11}{30} & \frac{323}{360} & \frac{251}{720} \end{pmatrix}$

The first characteristics polynomial of the block method of the discrete schemes in (11) is given by

$$p(\xi) = \det(\xi A_2^{(2)} - A_1^{(2)}) = 0$$
$$= \left|\xi A_2^{(2)} - A_1^{(2)}\right|$$
$$= 0$$

Now we have,

$$\rho(\xi) = \left| \xi \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| = \left| \begin{pmatrix} 0 & 0 & -\xi & 0 \\ \xi & 0 & -\xi & 0 \\ 0 & \xi & -\xi & 0 \\ 0 & 0 & -\xi & \xi \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right|$$

$$\rho(\xi) = \begin{vmatrix} 0 & 0 & -\xi & 1 \\ \xi & 0 & -\xi & 0 \\ 0 & \xi & -\xi & 0 \\ 0 & 0 & -\xi & \xi \end{vmatrix} = -\xi^4 + \xi^3 = 0$$

,

Using Maple (18) software

$\Rightarrow \xi_1 = 1$, $\xi_2 = 0$, $\xi_3 = 0$ and $\xi_4 = 0$

Since $|\xi_i| \le 1$, i = 1, 2, 3 and 4 then we observe that the discrete schemes in (11) satisfies the root condition and hence zero stable.

CONVERGENCE

The block discrete schemes methods in (11) are convergent since they are both consistent and zero-stable

STABILITY ANALYSIS

The stability analysis of numerical method for DDEs is considered. We considered on finding the P- and Q-stability of methods applied to the following DDEs of the form

$$y'(t) = \lambda y(t) + \mu y(t-\tau), \qquad t \ge t_0$$
$$y(t) = g(t), \qquad t \le t_0$$

(12)

where $g(t)$ is the initial function λ, μ are complex coefficients, $\tau = mh, m \in \mathbb{Z}^+$ and h is a step size or length. Let $H_1 = h\lambda$ and $H_2 = h\mu$, then from the discrete schemes in (11),

let $Y_{N+4} = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{pmatrix}, Y_N = \begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}, F_{N+4} = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix}$ and $F_N = \begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}$

Since $A_2^{(2)} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, A_1^{(2)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

$B_2^{(2)} = \begin{pmatrix} -\frac{51}{40} & -\frac{9}{10} & -\frac{21}{40} & \frac{3}{80} \\ -\frac{17}{45} & -\frac{19}{15} & -\frac{17}{45} & \frac{1}{90} \\ \frac{37}{360} & -\frac{19}{30} & -\frac{173}{360} & \frac{19}{720} \\ \frac{53}{360} & -\frac{11}{30} & \frac{323}{360} & \frac{251}{720} \end{pmatrix}$ and $B_1^{(2)} = \begin{pmatrix} 0 & 0 & 0 & -\frac{27}{80} \\ 0 & 0 & 0 & \frac{1}{90} \\ 0 & 0 & 0 & -\frac{11}{720} \\ 0 & 0 & 0 & -\frac{19}{720} \end{pmatrix}$

According to Sirisena *et al.* (2019), the P- and Q-stability polynomials are obtained by applying to (12). Thus the P-stability polynomials for the two discrete schemes in (11) are given respectively by

$$\psi^{(1)}(\xi) = \det\left[(A_2^{(1)} - H_1 B_2^{(1)})\xi^{2+m} - (A_1^{(1)} - H_1 B_1^{(1)})\xi^{1+m} - H_2 \sum_{i=1}^{2} B_i^{(1)} \xi^i \right]$$

and

$$\psi^{(2)}(\xi) = \det\left[(A_2^{(2)} - H_1 B_2^{(2)})\xi^{2+m} - (A_1^{(2)} - H_1 B_1^{(2)})\xi^{1+m} - H_2 \sum_{i=1}^{2} B_i^{(2)} \xi^i \right].$$

Also the Q-stability polynomials for the two discrete schemes in (3.15) are given respectively by

$$\pi^{(1)}(\xi)=\det\left[A_2^{(1)}\xi^{2+m}-A_1^{(1)}\xi^{1+m}-H_2\sum_{i=1}^{2}B_i^{(1)}\xi^i\right]$$

and

$$\pi^{(2)}(\xi)=\det\left[A_2^{(2)}\xi^{2+m}-A_1^{(2)}\xi^{1+m}-H_2\sum_{i=1}^{2}B_i^{(2)}\xi^i\right].$$

REGION OF ABSOLUTE STABILITY

The P-stability and Q-stability, is to show the discrete schemes which can be used with or without restriction on step size because of stability, and since $m=\frac{\tau}{h}$, m increases as h decreases. Hence, the important case is to show that the method is P-stable or Q-stable for value of m = 1. For each of the following discrete schemes in (11), we plot the region of P- and Q-stability for value of $m=1$.

Using Maple and MATLAB the P- and Q-stability region for the schemes in (11) are shown in Fig. 1 to Fig. 2

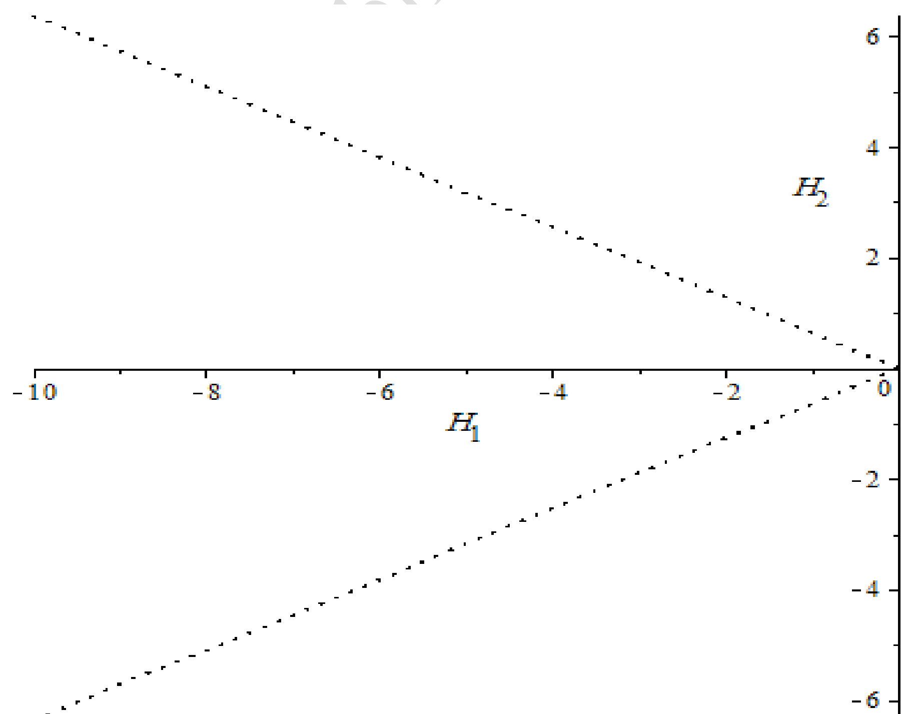

Fig.1 The P-stability region of the schemes in (11)

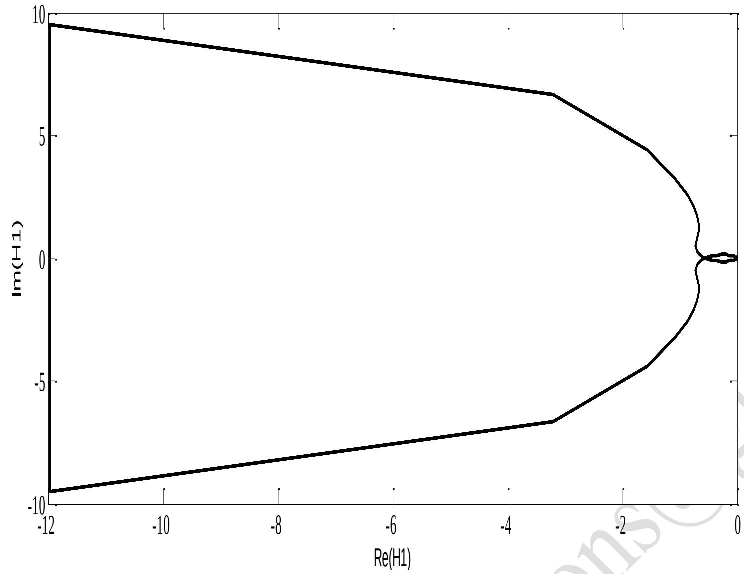

Fig.6 The Q-stability region of the schemes in (3.15)

NUMERICAL RESULTS

In this section, some Delay Differential Equations shall be solved using the block form of the discrete schemes derived in the above section. The delay term is implemented using the formula in Sirsena *et. al*. (2019)

NUMERICAL PROBLEMS

Example 1

$$y'(t) = -1000y(t) + 997e^{-3}y(t-1) + (1000 - 997e^{-3}), \qquad 0 \le t \le 1.5$$

$$y(t) = 1 + e^{-3t}, t \le 0$$

Exact solution $y(t) = 1 + e^{-3t}$

Solution:

Substituting, $h = 0.01, f = -1000y(t) + 997e^{-3}y(t-1) + (1000 - 997e^{-3})$ and corresponding values of f_n's using the discrete schemes in (3.15) .The results of the above are obtained in block form using Maple 18 varying

$n = 0, 1, 2, \ldots, 298$ and evaluating the values of y_n . The results are summarized in the table 1

Table 1. Solution of example 1

t	Exact Solution	Numerical Solution
0.1	1.740818221	1.740818222
0.2	1.548811636	1.548811635
0.3	1.406569660	1.406569658
0.4	1.301194212	1.301194215
0.5	1.223130160	1.223130162
0.6	1.165298888	1.165298887
0.7	1.122456428	1.122456429
0.8	1.090717953	1.090717953
0.9	1.067205513	1.067205514
1.0	1.049787068	1.049787067
1.1	1.036883167	1.036883166
1.2	1.027323722	1.027323722
1.3	1.020241911	1.020241911
1.4	1.014995577	1.014995574
1.5	1.011108997	1.011108998

Example 2

$$y'(t) = -1000y(t) + y(t - (\ln(1000-1))), \qquad 0 \le t \le 1.5$$

$$y(t) = e^{-t}, t \le 0$$

Exact solution $y(t) = e^{-t}$

Solution:

Substituting, $h = 0.01$, $f = -1000y(t) + y(t - (\ln(1000-1)))$ and corresponding values of f_n's using the discrete schemes in (3.15) .The results of the above are obtained in block form using Maple 17 varying

$n = 0, 1, 2, \ldots, 298$ and evaluating the values of y_n . The results are summarized in the table 2

Table 2 Solution of example 2

t	Exact Solution	Numerical Solution
0.1	0.904837418	0.904837332
0.2	0.818730753	0.818730626

0.3	0.740818221	0.740818149
0.4	0.670320046	0.670319933
0.5	0.606530660	0.606530600
0.6	0.548811636	0.548811547
0.7	0.496585304	0.496585255
0.8	0.449328964	0.449328893
0.9	0.406569660	0.406569619
1.0	0.367879441	0.367879381
1.1	0.332871084	0.332871051
1.2	0.301194212	0.301194163
1.3	0.272531793	0.272531766
1.4	0.246596964	0.246596924
1.5	0.223130160	0.223130138

Example 3

$y'(t) = -24y(t) - e^{(-25)}y(t-1), \qquad 0 \le t \le 1.5$

$y(t) = e^{(-25)t}, t \le 0$

Exact solution $y(t) = e^{(-25)t}$

Solution:

Substituting, $h = 0.01, f = -24y(t) - e^{(-25)}y(t-1)$ and corresponding values of $f_n{'s}$ using the discrete schemes in (3.15) .The results of the above are obtained in block form using Maple 17 varying

$n = 0, 1, 2, \ldots, 298$ and evaluating the values of y_n . The results are summarized in the table 3

Table 3. Solution of example 3

t	Exact Solution	Numerical Solution
0.1	0.082084999	0.082085411
0.2	0.006737947	0.006738062
0.3	0.000553084	0.000553098
0.4	4.53999E-05	4.54017E-05
0.5	3.72665E-06	3.72683E-06

0.6	3.05902E-07	3.05922E-07
0.7	2.51100E-08	2.51119E-08
0.8	2.06115E-09	2.06135E-09
0.9	1.69190E-10	1.69208E-10
1.0	1.38879E-11	1.38898E-11
1.1	1.13999E-12	1.14016E-12
1.2	9.35762E-14	9.35930E-14
1.3	7.6812oE-15	7.68276E-15
1.4	6.30512E-16	6.30660E-16
1.5	5.17556E-17	5.17693E-17

Example 4

$y'(t) = -y(t) - \frac{\pi}{2} e^{-1} y(t-1), \qquad 0 \le t \le 1.5$

$y(t) = e^{-t} \sin(\frac{\pi}{2} t), t \le 0$

Exact solution $y(t) = e^{-t} \sin(\frac{\pi}{2} t)$

Solution:

Substituting, $h = 0.01$, $f = -y(t) - \frac{\pi}{2} e^{-1} y(t-1)$ and corresponding values of f_n's using the discrete schemes in (3.15) .The results of the above are obtained in block form using Maple 18 varying

$n = 0, 1, 2, \ldots, 298$ and evaluating the values of y_n . The results are summarized in the table 4.

Table 4. Solution of example 4

T	Exact Solution	Numerical Solution
0.1	0.141547758	0.141547758
0.2	0.253001717	0.253001717
0.3	0.336324435	0.336324435
0.4	0.394004238	0.394004239

0.5	0.428881943	0.428881944
0.6	0.443997941	0.443997942
0.7	0.442460746	0.442460748
0.8	0.427337240	0.427337242
0.9	0.401564113	0.401564115
1.0	0.367879441	0.367879445
1.1	0.328772888	0.328772891
1.2	0.286452718	0.286452722
1.3	0.242827605	0.242827609
1.4	0.199501134	0.199501138
1.5	0.157776849	0.157776853

ANALYSIS OF RESULTS

The analysis of results is obtained by evaluating absolute difference of exact solutions and numerical solutions. The results are summarized in the tables 5 to 8

Table 5. Absolute errors of the method using Example 1

t	Error
0.1	1.32E-09
0.2	1.09E-09
0.3	1.74E-09
0.4	3.09E-09
0.5	1.85E-09
0.6	1.22E-09
0.7	7.47E-10
0.8	2.89E-10
0.9	1.26E-09
1.0	1.37E-09
1.1	1.40E-09
1.2	4.47E-10
1.3	4.46E-10
1.4	2.82E-09

1.5	1.46E-09

Table 6. Absolute error of the method using Example 2

t	Error
0.1	8.56E-08
0.2	1.27E-07
0.3	7.20E-08
0.4	1.13E-07
0.5	6.01E-08
0.6	8.96E-08
0.7	4.84E-08
0.8	7.16E-08
0.9	4.06E-08
1.0	5.98E-08
1.1	3.28E-08
1.2	4.85E-08
1.3	2.69E-08
1.4	4.03E-08
1.5	2.21E-08

Table 7. Absolute errors of the method using Example 3

t	Error
0.1	4.12E-07
0.2	1.15E-07
0.3	1.32E-08
0.4	1.72E-09
0.5	1.75E-10
0.6	1.94E-11
0.7	1.89E-12
0.8	1.95E-13
0.9	1.85E-14
1.0	1.84E-15
1.1	1.73E-16
1.2	1.67E-17
1.3	1.56E-18

1.4	1.49E-19
1.5	1.37E-20

Table 8. Absolute errors of the method using Example 4

t	Error
0.1	1.25E-10
0.2	1.18E-10
0.3	4.19E-11
0.4	6.29E-10
0.5	5.32E-10
0.6	1.47E-09
0.7	1.80E-09
0.8	2.76E-09
0.9	2.55E-09
1.0	3.33E-09
1.1	3.29E-09
1.2	3.95E-09
1.3	3.97E-09
1.4	4.06E-09
1.5	4.03E-09

CONCLUSIONS

In conclusion, the discrete schemes of the method respectively were deduced from their respective continuous formulations.

It was observed that, the block schemes are convergent, P-stable and Q-stable.

It was also observed in tables 1 to 4 that the performed better when compare with exact solutions

RECOMMENDATIONS

It is also recommended that the method is efficient for solving DDEs.

REFERENCES

Heng, S.C., Ibrahim, Z.B., Suleiman and Ismail. (2013). Solving Delay Differential Equations by using Implicit 2-Point Block Backward Differentiation Formulae. *Pentica J. Sci & Technol.* 21(1): 37 - 44

Ismail F., Al-Khasawneh R.A, Lwin A.S. & Suleiman M.B. (2002). Numerical treatment of delay differential equations by Runge-Kutta method using Hermite interpolation. *Matematika* 18: 79-90.

Ishak F., Suleiman M.B. & Omar Z. (2008). Two-point predictor-corrector block method for solving delay differential equations. *Matematika* 24 (2): 131-140

Ishak, F., Majid, Z. A and Suleiman, M. (2010). Two-point block method in variable step size technique for solving delay differential equations, *Journal of Materials Sc. and Eng*., vol. 4, no. 12, pp. 86–90

Ishak F., Majid Z.A & Suleiman M.B. (2013). Efficient interpolators in implicit block method for solving delay differential equations. *International Journal of Mathematics and Computers in Simulation.*

Karline S., Jeff C. &Francesca M. (2012). *Solving Differential Equations in R*, Springer Heidelberg, New York Dordrecht London. pp. 117–135.

Lumb, P. M. (2004). *Delay differential equations: Detection of small solutions*. (Unpublished doctoral dissertation). University of Liverpool, United Kingdom.

Majid Z.A., Suleiman M.B. & Omar Z. (2006). 3-point implicit block method for solving ordinary differential equations. *Bulletin of the Malaysian Mathematical Sciences* 29(1): 23–31

Majid Z.A., Radzi H.M & Ismail F. (2013). Solving delay differential equations by the five-point one-step block method using Neville's interpolation. *International Journal of Computer Mathematics*, http://dx.doi.org/10.1080/00207160.2012.754015

May, R. M. (1975). *Stability and complexity in model ecosystems (2nd Ed.)*. Princeton: Princeton University Press

Mohamed A.R, Abd-El-Aziz E.S. & Mahmoud N.S. (2006). Numerical solution of system of first-order delay differential equations using polynomial spline functions. *International Journal of Computer Mathematics* 83(12): 925–937

Oberle H.J. & Pesh H.J. (1981). Numerical treatment of delay differential equations by Hermite interpolation. *Numer. Math* 37: 235–255.

Radzi, H. M, Majid, Z. A, Ismail, F. & Suleiman, M. (2012). Two and three point one-step block methods for solving delay differential equations. *Journal of Quality Measurement and Analysis* 82(1), 29–41

Rosser, J.B. (1967), A Runge–Kutta for all seasons, *SIAM Rev*. 9 pp. 417–452.

San, H.C. Majid, Z.A. & Othman, M. (2011). Solving delay differential equations using coupled block method, *The 4th International Conference on Modeling, Simulation and Applied Optimization (ICMSAO)*, Kuala Lumpur, pp. 1–4.

Sirisena, U. W. (1997). *A Reformulation of the Continuous General Linear Multistep Method by Matrix Inversion for the First Order Initial Value Problems*. Ph.D. Thesis (Unpublished), University of Ilorin, Nigeria

Sirisena, U. W. & Yakubu S.Y. (2019). Solving Delay Differential Equation using Reformulated Backward Differentiation Methods. *British Journal of mathematics & Computer Science* 32(2) 1 – 15.