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# ALGORITHM TO SIMPLIFY ISSUES IN LINEAR DIFFERENCE EQUATIONS

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**ABSTRACT**

*This paper explains the algorithms of analyzing finite difference equations, for the solution of ordinary difference equations, to reduce the time of computation based on its theory and numerical illustrations were used for clarity.*

*Keywords: Simplify, Issues, Linear, Difference, Equations.*

**Introduction:**

A difference equation is an equation involving an independent variable. Therefore, it may also defined as an equation which expresses a relation between an independent variable and the successive value of the dependent variable.

When solving linear difference equation with constant coefficients one first finds the general solution for the homogenous solution to the non – homogenous one. The same recipe works in the case of difference equations, ie first find the general solution to $\phi(E)U_n = 0$ and a particular solution to $\phi(E) = f(n)$ adding the two together for the general solution to the latter equation. Thus to solve these more general equations, the only new problem is to identify some particular solution

## Methodology

A linear difference equation of first –order which have the form $y_{k+1} = ay_k + b \forall k = 0,1,2,3...$ Where a and b are given function of k will be solve and will serve as the background for the modeling.

In searching for closed form solutions, we assume that both (a) and (b) are constants. The difference equation is said to be homogenous if b is zero and constant coefficient if 'a' is a constant. The homogenized equation is the difference equation $x_{k+1} = ax_k \forall \quad k = 0,1,2,3...$

**Finite Linear Difference Equation without constant coefficient**

With constant coefficient consider a finite difference equation of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = \phi_n \text{ ---------------------------------- (1)}$$

Where $\alpha_j \, j = 0(1)k$ are constant and $\emptyset_n$ is a constant. Usually equation (1) is called a finite linear equation difference equation with constant coefficient . generally, we shall need k initial conditions to ensure that the equation has a solution, then our interest is to solve the difference equation given below.

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = \phi_n, y_0 = \tau_1, y_1 = \tau_2, y_k = \tau_k \text{ --------(2)}$$

The general solution to equation (2) is given by $y_n = y_n H + y_n p$ where $y_n H$ is the solution to the homogenous (LHS) of 2 and $y_{n,p}$ is the particle solution (RHS) of (2). The equation (2) above can be written as

$$\alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + ---- \alpha_k y_{n+k} = \phi_n$$

With initial condition specified the particular solution to (2) is given by

$$y_{n,p} = \frac{\phi_n}{\sum_{j=0}^{k} \alpha_j} \quad \text{provided} \quad \sum_{j=0}^{k} \alpha_j \neq 0$$

The homogenous part of (2) is given by $\sum_{j=0}^{k} \alpha_j y_{n+j} = 0 \text{------------(3)}$
ie equation right hand side to zero

**Finite Linear Difference Equation with Constant Coefficient**

Consider a finite difference equation of form

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$$\sum_{j=0}^{k}\alpha_j y_{n+j}=\phi_n$$ ---------------------------------(1b)

Where $\alpha_j, j=0$ (i)k are constants and $\phi_n$ is a constant, usually equation (1b) is called a finite linear difference equation with constant coefficient. Generally we should need k initial condition to ensure that the equation(1b) has a solution thus, our interest to solve the difference equation given below

$$\sum_{j=0}^{k}\alpha_j y_{n+j}=\phi_n, y_0=\tau_1, y_1=\tau_2 ----y_{k-1} ---------------(2b)$$

The general solution to equation (2b) is given

$$y_n = y_{n,H} + Y_{n.p}$$

Where $y_{n,H}$ is the solution to the homogenous (LHS) of (2b) and $y_{n,p}$ is the particular solution (RHS) of (2b) .

The equation(2b) above can be written as

$$\alpha_o y_n + \alpha y_{n+1} + \alpha_2 y_{n+2} + ----------- + \alpha_k y_{n+k} = \phi_n$$

Where $y_{n,H}$ is the solution to the homogenous (LHS) of (2b) and $y_{n,p}$ is the particular solution (RHS) of (2b).

The equation(2b) above can be written as; $\alpha_o y_n + \alpha y_{n+1} + \alpha_2 y_{n+2} + ----------- + \alpha_k y_{n+k} = \phi_n$ with initial condition specified the particular, solution (2b) is given $y_{n,p}=\dfrac{\phi_n}{\sum_{j=0}^{k}\alpha_j}$ provide $\sum_{j=0}^{k}\alpha_j \neq 0$

The homogenous part (2) is given by $\sum_{j=0}^{k}\alpha_j y_{n+j}=0$ Equate RHS to Zero

**Algorithm**

The solution to (3) is obtained as follows;

**Step 1** Form a polynomial of degree k in r $p_k(r)$ by replacing $y_{n+j}$ by $r^{n+j}$

$\sum_{j=0}^{k}\alpha_j r^{n+j}=0$ is set $y_{n+j}$ to $r^{n+j}$

Hence,

$$r^n\sum_{j=0}^{k}\alpha_j r^j=0$$

$\sum_{j=0}^{k}\alpha_j r^j=0$ is divided through by $r^n$ which will yield k roots

**Step 2** Obtain the roots $p_k(r)\sum_{j=0}^{k}\alpha_j r^j=0$

Nature of roots

(a) If $r_1, r_2 ----r_n$ are real and distinct roots of $p_k(r)$ then the homogenous solution is given by

$y_{nH}=\sum_{i=1}^{k}\beta_1 r_i^n$ where $\beta_1$, i=1(i)k are constants to be determined using the initial value given in equation (2)

(b) If $r_1=r_2=r_3----r_k$ where the roots are repeated i.e $p_k(r)$ has multiplicity of roots. Then the homogenous solution is given by

$$y_{n,H}=\beta_1 r_1^n+n\beta_2 r_1^n+n(n-1)\beta_3 r_1^n+n(n-1)(n-2)\beta_4 r^n+\ldots$$

(c) If in case (b) above, three of the roots are repeated while the others are distinet then

$$y_{n,H}=\beta_1 r_1^n+n\beta_2 r_1^n+n(n-1)\beta_3 r_1^n+\sum_{i=4}^{k}\beta_i r_i^n$$

So that for case (a), the general solution to (2) is given by

$$y_n=y_{n,h}+y_{n,H}=\sum_{i=1}^{k}\beta_i r_i^n+\frac{\phi}{\sum_{j=0}^{k}\alpha_j}$$

And for case (b), we have

$$y_{n,H}=\beta_1 r_1^n+n\beta_2 r_1^n+n(n-1)\beta_3 r_1^n+n(n-1)(n-2)\beta_4 r^n+\ldots+\frac{\phi_n}{\sum_{j=0}^{k}\alpha_j}$$

**Step 3**

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The solution to (3) is obtained as follow; (1) form a polynomial of degree k is r is $p_k(r)$ by replacing $y_{nrj}$ by $y_{nrj}$ by $r^{n+j}$

$\sum_{j=0}^{k}\alpha_j r^{n+j}=0$ is set $y_{n+j}$ to $r^{n+j}$

Hence $r^n\sum_{j=0}^{k}\alpha_j r^j=0$

$\sum_{j=0}^{k}\alpha_j r^j=0$ is divided through by $r^n$ which will yield k – roots

(a) Obtain the root $p_k(r)=\sum_{j=0}^{k}\alpha_j r^j=0$

**Nature of roots**

If $r_1, r_2 ---r_k$ are real and distinct roots of $p_k(r)$ then the homogenous solution is given by

$y_{n,H}=\sum_{i=1}^{k}\beta_i r_i^n$ where $\beta_i$ i=1(1)k are constants to be determined using the initial values given in equation (2)

(b) If $r_1=r_2=r_3=----=r_k$ where the roots are repeated in $p_k(r)$ has small multiplicity of roots when the homogenous is given by;

$$y_{n,H}=\beta_1 r_1^n+n\beta_2 r_1^n+n(n-1)\beta_3 r_1^n+n(n-1)(n-2)\beta_4 r^n+\ldots$$

(c) if case (b) above three, three of the roots are repeated while the other are distinct. Then

$$y_{n,H}=\beta_1 r_1^n+n\beta_2 r_1^n+n(n-1)\beta_3 r_1^n+\sum_{i=4}^{k}\beta_i r_i^n$$

So that for case (a), the general solution to (2) is given by

$y_n=y_{n,p}+y_{n,H}=\sum_{i=1}^{k}\beta_i r_i^n+\frac{\phi}{\sum_{j-0}^{k}\alpha_{j\partial}}$ and for case (b) we have

$$y_{n,H}=\beta_1 r_1^n+n\beta_2 r_1^n+n(n-1)\beta_3 r_1^n+n(n-1)(n-2)\beta_4 r^n+....+\frac{\phi}{\sum_{j=0}^{k}\alpha_j}$$

For case (c)

$$y_{n,H}=\beta_1 r_1^n+n\beta_2 r_1^n+n(n-1)\beta_3 r_1^n+n(n-1)(n-2)\beta_4 r^n+\sum_{i=4}^{k}\beta_i r_i^n+\frac{\phi}{\sum_{j=0}^{k}\alpha_j}$$

**Numerical Illustration (1)**

Solve the difference equation

$y_{n+1}-3y_n=5, y_0=0$

Solution

$y_n=y_{n,p}+y_{n,H}$ Where

$$y_{n,p}=\frac{\phi}{\sum_{j=0}^{k}\alpha_j}=\frac{5}{1-3}=\frac{5}{2}$$

And $y_{n,H}$ set

$y_{n+1}-3y_n=0$

Form a polynomial of r by setting

$y_{n+1}=r^{n+1}$ and $y_n=r^n$

Hence,

$r^{n+1}-3r^n=0$

$r^n.r-3r^n=0$

$r^n(r-3)=0$ divide through by $r^n$

r-3 =0 therefor r=3

Hence,

$$y_{n,H}=\sum_{j=0}^{k}\beta_i r_i^n=\sum_{i=1}^{k}\beta_i r_1^n=\beta_1 r_i^n=\beta_1(3)^n$$

$\therefore\ y_n-y_{n,p}+y_{n,H}=\beta_1 3^n-\frac{5}{2}$

To determine the value of $\beta_1$ we use the initial condition $y_o=0$

$$y_n = \beta_1 3^n - \frac{5}{2}$$

$$y_0 = \beta_1 3^n - \frac{5}{2}$$

$$0 = \beta_1 - \frac{5}{2}$$

$$\beta_1 = \frac{5}{2}$$

$$y_n = \frac{5}{2}3^n - \frac{5}{2}$$

$$y_n = \frac{5}{2}\left(3^n - 1\right)$$

## Numerical Illustration (2)

Solve the difference equation

$$y_{n+2} - y_{n+1} - y_n = 0, y_0 = 0, y_1 = 1$$

$$y_n = y_{n,H} + y_{n,p}$$

$$y_{n,p} = \frac{\phi}{\sum_{j=0}^{k} \alpha}$$

$$y_{n+2} = r_{n+2}$$

$$y_{n+1} = r_{n+1}$$

$\forall y_{nH}$ set $y_n = r^n$

Form a polynomial $p_k(r) = p_2(r)$

$$r^{n+2} - r^{n+1} - r^n = 0$$

$$r^n.r^2 - r^n.r^1 - r^n = 0$$

$$r^n(r^2 - r - 1) = 0$$

$$r^2 - r - 1 = 0$$

Since the roots are distance

i.e $r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$, then

$$y_{n,H} = \sum_{i=1}^{2} \beta_1 r_i^n = \beta_1 r_1^n + \beta_2 r_2^n$$

$$y_{n+1} = \left(\frac{1+\sqrt{5}}{2}\right)^n + \beta_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$y_n = y_{n,p} + y_{n,H} = 0 + \beta_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \beta_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

To determine $\beta_1$ and $\beta_2$, use the initial value conditions

$$y_0 = 0, y_1 = 1$$

$$y_0 = \beta_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + \beta_2 \left(\frac{1-\sqrt{5}}{2}\right)^0$$

$$0 = \beta_1 + \beta_2$$

$$\beta_1 = -\beta_2$$

When n=1 $y_1$=1

$$y_1 = \beta_1 \left(\frac{1+\sqrt{5}}{2}\right)^1 + \beta_2 \left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$

$$\beta_1 \left(\frac{1+\sqrt{5}}{2}\right) + \beta_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$\beta_1 \left(\frac{1+\sqrt{5} - 1 - \sqrt{5}}{2}\right) = 1$$

$$\beta_1 \left(\frac{2\sqrt{5}}{2}\right) = 1$$

$$\beta_1 = \frac{1}{\sqrt{5}}, \beta_2 = \frac{-1}{\sqrt{5}}$$

$$y_1 = \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^1 - \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^1 = 1$$


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$$y_n = \left[ \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n + \beta_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \right]$$

The most common form of a linear difference equation may be $y_{n+1} + p_1 y_{x+n-1} + p_2 p_{x+n-2} + p_n y_x = \beta(x)$

where $p_1, p_2 ----p_n$ may be absolute constants or periodic function of x. $\beta$ (x) is some function of the independent variable x.
If B(x) is zero, the linear difference equation will be called linear homogenous difference equation of order n. if B(x)≠0, it will be called linear non homogenous or completed difference equation of order n.

$y_{x+n} + p_1 y_{x+n-1} + ------- + p_n p_x = 0 ------(4)$

Where are $p_i^{'} s$ functions of x

Analogue to the theorems of ordinary differential results which may however be proved easily

1. If $y_i(x)$ is a solution of equation (4) so is $c_1 y_1(x)$ where $c_1$ is an arbitrary constant
2. if $y_1(x), y_2(x), ---- y_n(x)$ are distinct solutions of equation (4) then $c_1 y_1(x) + c_2 y_2 + \cdots + c_n y_x(x)$ is a solution

Since linear equation with variable coefficients are solvable under some restriction on the coefficients. We considered at the beginning, linear equation with constants coefficients which may be written in the form $y_{x+n} + a_1 y_{x+n} + \cdots a_n y_x = 0$
using operator E, we have

$$[E^n + aE^{n-1} + a_{n-1}E + a_n]y_x = 0 ------(5)$$

Equation (5) may be written as $f(E) \equiv E^n + a_1 E^{n-1} + \cdots + a_{n-1}E + a_n$ and the coefficients $a_1 . \; . \; . a_{n-1}, a_n$ are constants. We shall call $f(E) \equiv E^n + a_1 E^{n-1} + \cdots + a_n$ is known as the characteristics function of the given equation.

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